

GREEN'S FORMULA AND A DIRICHLET-TO-NEUMANN OPERATOR FOR FRACTIONAL-ORDER PSEUDODIFFERENTIAL OPERATORS

GERD GRUBB

Department of Mathematical Sciences, Copenhagen University, Universitetsparken 5, DK-2100 Copenhagen, Denmark. E-mail grubb@math.ku.dk

ABSTRACT. The paper treats boundary value problems for the fractional Laplacian $(-\Delta)^a$, $a > 0$, and more generally for classical pseudodifferential operators (ψ do's) P of order $2a$ with even symbol, applied to functions on a smooth subset Ω of \mathbb{R}^n . There are several meaningful local boundary conditions, such as the Dirichlet and Neumann conditions $\gamma_k^{a-1}u = \varphi$, $k = 0, 1$, where $\gamma_k^{a-1}u = c_k \partial_n^k(u/d^{a-1})|_{\partial\Omega}$, $d(x) = \text{dist}(x, \partial\Omega)$. We show a new Green's formula

$$(Pu, v)_\Omega - (u, P^*v)_\Omega = (s_0\gamma_1^{a-1}u + B\gamma_0^{a-1}u, \gamma_0^{a-1}v)_{\partial\Omega} - (s_0\gamma_0^{a-1}u, \gamma_1^{a-1}v)_{\partial\Omega},$$

where B is a first-order ψ do on $\partial\Omega$ depending on the first two terms in the symbol of P .

Moreover, we show in the elliptic case how the Poisson-like solution operator K_D for the nonhomogeneous Dirichlet problem is constructed from P^+ in the factorization $P \sim P^-P^+$ obtained in earlier work. The Dirichlet-to-Neumann operator $S_{DN} = \gamma_1^{a-1}K_D$ is derived from this as a first-order ψ do on $\partial\Omega$, with an explicit formula for the symbol. This leads to a characterization of those operators P for which the Neumann problem is Fredholm solvable.

1. Introduction.

The fractional Laplacian $(-\Delta)^a$ on \mathbb{R}^n , $0 < a < 1$, and its boundary value problems on subsets $\Omega \subset \mathbb{R}^n$, have received much attention recently, as a generalization from the ordinary Laplacian $-\Delta$, with useful applications to probability, finance, differential geometry and mathematical physics. $(-\Delta)^a$ can be described as a pseudodifferential operator (ψ do) or as a singular integral operator:

$$\begin{aligned} (-\Delta)^a u &= \text{OP}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)) \\ (1.1) \quad &= c_{n,a}PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|y|^{n+2a}} dy. \end{aligned}$$

The discussion we give in this paper works for classical ψ do's P of order $2a$ (any $a > 0$) with symbol $p \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi)$ being *even*:

$$(1.2) \quad p_j(x, -\xi) = (-1)^j p_j(x, \xi), \text{ all } j.$$

E.g. $P = A(x, D)^a$, where $A(x, D)$ is a second-order strongly elliptic differential operator.

One of the difficulties with these operators is that they are *nonlocal*, in contrast to differential operators. This is problematic when one wants to study them on subsets $\Omega \subset \mathbb{R}^n$.

It has been known for many years that one can define a Dirichlet realization P_D by a variational construction: Define the sesquilinear form $p_0(u, v)$ by

$$(1.3) \quad p_0(u, v) = \int_{\Omega} Pu \bar{v} dx, \quad u, v \in C_0^\infty(\Omega),$$

completed to a form on $\dot{H}^a(\bar{\Omega}) = \{u \in H^a(\mathbb{R}^n) \mid \text{supp } u \subset \bar{\Omega}\}$. It is coercive when P is strongly elliptic, and then induces an operator P_D in $L_2(\Omega)$ acting like r^+P with domain

$$(1.4) \quad D(P_D) = \{u \in \dot{H}^a(\bar{\Omega}) \mid r^+Pu \in L_2(\Omega)\}, \quad r^+Pu = (Pu)|_{\Omega};$$

a Fredholm operator when Ω is bounded. P_D represents the (fractional restricted) homogeneous Dirichlet problem

$$(1.5) \quad r^+Pu = f \text{ in } \Omega, \quad \text{supp } u \subset \bar{\Omega}.$$

Vishik and Eskin treated such operators in the 1960's (see e.g. [E81]) by pseudodifferential factorization methods; one consequence is that $D(P_D) = \dot{H}^{2a}(\bar{\Omega})$ when $a < \frac{1}{2}$, and $D(P_D) \subset \dot{H}^{a+\frac{1}{2}-\varepsilon}(\bar{\Omega})$ when $a \geq \frac{1}{2}$.

More recently, real integral operator methods have been applied. There is a technique by Caffarelli and Silvestre [CS07] to view $(-\Delta)^a$ on \mathbb{R}^n as the Dirichlet-to-Neumann operator for a degenerate elliptic differential boundary value problem on $\mathbb{R}^n \times \mathbb{R}_+$, which allows local techniques. For subsets $\Omega \subset \mathbb{R}^n$, methods from potential theory and probability have particularly been used, leading to results in Hölder spaces, under low smoothness assumptions. Also functional analysis methods enter. Let us mention some of the studies through the times: Blumenthal and Gettoor [BG59], Landkof [L72], Hoh and Jacob [HJ96], Kulczycki [K97], Chen and Song [CS98], Jakubowski [J02], Silvestre [S07], Caffarelli and Silvestre [CS09], Frank and Geisinger [FG16], Ros-Oton and Serra [RS14a, RS14b, RS16], Abatangelo [A15], Felsinger, Kassmann and Voigt [FKV15], Bonforte, Sire and Vazquez [BSV15], Servadei and Valdinoci [SV14], Ros-Oton [R16]; there are many more papers referred to in these works, and numerous applications to nonlinear problems.

To the question of regularity of solutions, Ros-Oton and Serra [RS14a] obtained for (1.5) with $P = (-\Delta)^a$ that $f \in L_\infty$ implies $u \in d^\alpha C^\alpha(\bar{\Omega})$ for small α ; here $d(x) = \text{dist}(x, \partial\Omega)$. This was improved later to $\alpha < a$, and the result was extended to more general translation-invariant singular integral operators with even kernels [RS16].

In very recent years, ψ do methods have come into the picture again, through works of the author [G14, G15, G16], based on the a -transmission property introduced by Hörmander [H65, H85], combined with methods from the Boutet de Monvel calculus [B71, G90, G96, G09]. This approach allows x -dependent operators. The systematic treatment of ψ do's primarily takes place in smooth situations.

It was shown in [G15] that the domain space for solutions of (1.5) with $f \in L_p(\Omega)$ ($1 < p < \infty$, Ω bounded smooth), equals $H_p^{a(2a)}(\bar{\Omega})$, where $H_p^{a(s)}(\bar{\Omega}) = \Lambda_+^{(-a)} e^+ \bar{H}_p^{s-a}(\Omega)$

(definitions are recalled in Section 2). For a nonhomogeneous Dirichlet problem, one must pass to the larger space $H_p^{(a-1)(s)}(\overline{\Omega})$, where the Dirichlet trace operator

$$(1.6) \quad \gamma_0^{a-1}u = \Gamma(a)\gamma_0(u/d^{a-1})$$

(denoted $\gamma_{a-1,0}u$ in [G14, G15, G16]) makes good sense for $s > a - 1/p'$, $\gamma_j u = (\partial_n^j u)|_{\partial\Omega}$. Also a Neumann trace operator is defined there, for $s > a + 1/p$,

$$(1.7) \quad \gamma_1^{a-1}u = \Gamma(a+1)\gamma_1(u/d^{a-1}).$$

If $\gamma_0^{a-1}u = 0$ then $u \in H_p^{a(s)}(\overline{\Omega})$ and $\gamma_1^{a-1}u = \gamma_0^a u$. Integration-by-parts formulas have up to now been shown involving only first boundary values [RS14b, A15, G16] (Corollary 4.5 below); they lead to a useful Pohozaev formula.

We can now for the first time show a full Green's formula for $u, v \in H^{(a-1)(s)}(\overline{\Omega})$ (Theorem 4.4 below):

$$(1.8) \quad \int_{\Omega} (Pu \bar{v} - u \overline{P^*v}) dx = \int_{\partial\Omega} (s_0 \gamma_1^{a-1}u \gamma_0^{a-1}\bar{v} - s_0 \gamma_0^{a-1}u \gamma_1^{a-1}\bar{v} + B \gamma_0^{a-1}u \gamma_0^{a-1}\bar{v}) d\sigma;$$

here $s_0(x)$ is the value of the principal symbol of P on the interior normal $\nu(x)$ at $x \in \partial\Omega$, $s_0(x) = p_0(x, \nu(x))$, and B is a first-order ψ do on $\partial\Omega$ (4.20), determined from the principal and subprincipal symbols of P . To our knowledge, a formula involving both nontrivial Neumann traces and nontrivial Dirichlet traces has not been found before. Ellipticity of P is not assumed.

Next, based on the factorization results of [G16], we establish in the elliptic case a detailed formula for the Poisson-like operator $K_D: \varphi \mapsto u$ solving the Dirichlet problem with nonhomogeneous boundary condition

$$(1.9) \quad r^+ Pu = 0 \text{ in } \Omega, \quad \gamma_0^{a-1}u = \varphi \text{ on } \partial\Omega,$$

in a parametrix sense, showing exactly how it arises from the plus-factor P^+ in $P \sim P^- P^+$ (Theorems 5.2, 5.4).

As an application, we describe the Dirichlet-to-Neumann operator (Theorem 6.1),

$$(1.10) \quad S_{DN} = \gamma_1^{a-1} K_D;$$

it is a first-order ψ do on $\partial\Omega$ with symbol derived from P^+ . Its principal symbol is, in local coordinates,

$$(1.11) \quad s_{DN,0}(x', \xi') = - \lim_{z_n \rightarrow 0^+} \mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \log(s_0^{-1} p_0(x', 0, \xi) |\xi|^{-2a}) - a |\xi'|.$$

(S_{DN} has only been described before in the elementary case of $P = (1 - \Delta)^a$ on \mathbb{R}_+^n , in [G14].) This implies a concrete characterization of the operators P for which the Neumann problem has Fredholm solvability (Theorem 6.2), namely those for which S_{DN} is elliptic ($s_{DN,0}$ is nonvanishing for $\xi' \neq 0$). Their parametrices are likewise described.

In view of [G14, G15], the parametrices and the Dirichlet-to-Neumann operator we have described, act in both H_p^s -spaces and in more general Besov and Triebel-Lizorkin spaces;

in particular in the Hölder-Zygmund spaces C_*^s that are of special interest for nonlinear applications (Corollary 6.4).

Plan of the paper: Section 2 gives preliminaries and notation, and the Appendix accounts for some results from the Boutet de Monvel calculus and [G16] that we use. In Section 3 we show Green's formula for the case of $P = (1 - \Delta)^a$ on \mathbb{R}_+^n , using only Fourier transformation and distribution theory. In Section 4, Green's formula is shown for general operators P and general smooth domains Ω ; this is based on reductions to applications of the Boutet de Monvel calculus and delicate localization techniques. Section 5 gives the construction of K_D , also using such tools. In Section 6, S_{DN} is derived, and applied to the discussion of solvable Neumann problems.

2. Preliminaries.

Multi-index notation is used for differentiation (and polynomials): $\partial = (\partial_1, \dots, \partial_n)$, and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ for $\alpha \in \mathbb{N}_0^n$, with $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$. $D = (D_1, \dots, D_n)$ with $D_j = -i\partial_j$.

The function $\langle \xi \rangle$ stands for $(1 + |\xi|^2)^{\frac{1}{2}}$, and $[\xi]$ denotes a positive C^∞ -function equal to $|\xi|$ for $|\xi| \geq 1$ and $\geq \frac{1}{2}$ for all ξ .

Operators are considered acting on functions or distributions on \mathbb{R}^n , and on subsets $\mathbb{R}_\pm^n = \{x \in \mathbb{R}^n \mid x_n \gtrless 0\}$ (where $(x_1, \dots, x_{n-1}) = x'$), and bounded C^∞ -subsets Ω with boundary $\partial\Omega$, and their complements. Restriction from \mathbb{R}^n to \mathbb{R}_\pm^n (or from \mathbb{R}^n to Ω resp. $\mathbb{C}\overline{\Omega}$) is denoted r^\pm , extension by zero from \mathbb{R}_\pm^n to \mathbb{R}^n (or from Ω resp. $\mathbb{C}\overline{\Omega}$ to \mathbb{R}^n) is denoted e^\pm . Restriction from $\overline{\mathbb{R}_+^n}$ or $\overline{\Omega}$ to $\partial\mathbb{R}_+^n$ resp. $\partial\Omega$ is denoted γ_0 . $\mathcal{S}(\overline{\mathbb{R}_+^n})$ stands for $r^+\mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is Schwartz' space of rapidly decreasing C^∞ -functions (with dual space $\mathcal{S}'(\mathbb{R}^n)$, the temperate distributions).

We denote by $d(x)$ a function of the form $d(x) = \text{dist}(x, \partial\Omega)$ for $x \in \Omega$, x near $\partial\Omega$, extended to a smooth positive function on Ω ; $d(x) = x_n$ in the case of \mathbb{R}_+^n . Then we define the spaces

$$(2.1) \quad \mathcal{E}_\mu(\overline{\Omega}) = e^+\{u(x) = d(x)^\mu v(x) \mid v \in C^\infty(\overline{\Omega})\},$$

for $\text{Re } \mu > -1$; for other μ , cf. [G15].

A *pseudodifferential operator* (ψ do) P on \mathbb{R}^n is defined from a symbol $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$(2.2) \quad Pu = p(x, D)u = \text{OP}(p(x, \xi))u = (2\pi)^{-n} \int e^{ix \cdot \xi} p(x, \xi) \hat{u} d\xi = \mathcal{F}_{\xi \rightarrow x}^{-1}(p(x, \xi) \hat{u}(\xi));$$

here \mathcal{F} is the Fourier transform $(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx$. We refer to textbooks such as Hörmander [H85], Taylor [T91], Grubb [G09] for the rules of calculus. [G09] moreover gives an account of the Boutet de Monvel calculus of *pseudodifferential boundary problems*, cf. also e.g. [G96]. We take p in the symbol space $S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$, consisting of C^∞ -functions $p(x, \xi)$ such that $\partial_x^\beta \partial_\xi^\alpha p(x, \xi)$ is $O(\langle \xi \rangle^{m-|\alpha|})$ for all α, β , for some $m \in \mathbb{R}$ (global estimates); then p and P have order m . A symbol p is said to be *classical* when it moreover has an asymptotic expansion $p(x, \xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi)$ with p_j homogeneous in ξ of degree $m - j$ for $|\xi| \geq 1$, all j , and $p(x, \xi) - \sum_{j < J} p_j(x, \xi) \in S_{1,0}^{m-J}(\mathbb{R}^n \times \mathbb{R}^n)$ for all J .

Recall in particular the composition rule: When $PQ = R$, then R has a symbol $r(x, \xi)$ with the following asymptotic expansion, called the Leibniz product:

$$(2.3) \quad r(x, \xi) \sim p(x, \xi) \# q(x, \xi) = \sum_{\alpha \in \mathbb{N}_0^n} \partial_\xi^\alpha p(x, \xi) D_x^\alpha q(x, \xi) / \alpha!.$$

When P is a ψ do on \mathbb{R}^n , $P_+ = r^+ P e^+$ denotes its truncation to \mathbb{R}_+^n , or to Ω , depending on the context.

Let $1 < p < \infty$ (with $1/p' = 1 - 1/p$), then the L_p -Sobolev spaces (Bessel-potential spaces) are defined for $s \in \mathbb{R}$ by

$$\begin{aligned} H_p^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\}, \\ \dot{H}_p^s(\overline{\Omega}) &= \{u \in H_p^s(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega}\}, \\ \overline{H}_p^s(\Omega) &= \{u \in \mathcal{D}'(\Omega) \mid u = r^+ U \text{ for some } U \in H_p^s(\mathbb{R}^n)\}; \end{aligned}$$

here $\text{supp } u$ denotes the support of u . The definition is also used with $\Omega = \mathbb{R}_+^n$. In most current texts, $\overline{H}_p^s(\Omega)$ is denoted $H_p^s(\Omega)$ without the overline (that was introduced along with the notation \dot{H} in [H65, H85]), but we keep it here since it is practical in indications of dualities, and makes the notation more clear in formulas where both types occur. When $p = 2$, the mention of p is left out. We recall that $\overline{H}_p^s(\Omega)$ and $\dot{H}_{p'}^{-s}(\overline{\Omega})$ are dual spaces with respect to a sesquilinear duality extending the $L_2(\Omega)$ -scalar product, written e.g.

$$\langle f, g \rangle_{\overline{H}_p^s(\Omega), \dot{H}_{p'}^{-s}(\overline{\Omega})}, \text{ or just } \langle f, g \rangle_{\overline{H}_p^s, \dot{H}_{p'}^{-s}}.$$

There are many other interesting scales of spaces, the Triebel-Lizorkin and Besov spaces $F_{p,q}^s$ and $B_{p,q}^s$, where the problems can be studied; see details in [G14]. This includes the Hölder-Zygmund spaces $B_{\infty,\infty}^s$, also denoted C_*^s ; they are interesting because $C_*^s(\mathbb{R}^n)$ equals the Hölder space $C^s(\mathbb{R}^n)$ when $s \in \mathbb{R}_+ \setminus \mathbb{N}$.

Hörmander introduced in [H65, H85] the μ -transmission condition at $\partial\Omega$ for a classical ψ do of order m , $\mu \in \mathbb{C}$: In local coordinates,

$$(2.4) \quad \partial_x^\beta \partial_\xi^\alpha p_j(x, -\nu) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, \nu),$$

for all $x \in \partial\Omega$, all j, α, β , where ν denotes the interior normal to $\partial\Omega$ at x . The Boutet de Monvel calculus treats the case where $\mu = 0$, and the operators P we shall study satisfy it with $\mu = a > 0$.

A special role in the theory is played by the *order-reducing operators*. There is a simple definition of operators Ξ_\pm^t on \mathbb{R}^n , $t \in \mathbb{R}$,

$$(2.5) \quad \Xi_\pm^t = \text{OP}(\chi_\pm^t), \quad \chi_\pm^t = (\langle \xi' \rangle \pm i\xi_n)^t;$$

they preserve support in $\overline{\mathbb{R}}_\pm^n$, respectively. The functions $(\langle \xi' \rangle \pm i\xi_n)^t$ do not satisfy all the estimates required for the class $S_{1,0}^t(\mathbb{R}^n \times \mathbb{R}^n)$, but the operators are useful for many purposes. There is a more refined choice Λ_\pm^t [G90, G15], with symbols $\lambda_\pm^t(\xi)$ that do satisfy all the estimates for $S_{1,0}^t(\mathbb{R}^n \times \mathbb{R}^n)$; here $\overline{\lambda}_+^t = \lambda_-^t$. The symbols have holomorphic

extensions in ξ_n to the complex halfspaces $\mathbb{C}_\mp = \{z \in \mathbb{C} \mid \operatorname{Im} z \leq 0\}$, and hence the operators preserve support in $\overline{\mathbb{R}}_\pm^n$, respectively; operators with that property are called "plus" resp. "minus" operators. There is also a pseudodifferential definition $\Lambda_\pm^{(t)}$ adapted to the situation of a smooth domain Ω , cf. [G15].

It is elementary to see by the definition of the spaces $H_p^s(\mathbb{R}^n)$ in terms of Fourier transformation, that the operators define homeomorphisms for all s :

$$(2.6) \quad \Xi_\pm^t : H_p^s(\mathbb{R}^n) \xrightarrow{\sim} H_p^{s-t}(\mathbb{R}^n), \quad \Lambda_\pm^t : H_p^s(\mathbb{R}^n) \xrightarrow{\sim} H_p^{s-t}(\mathbb{R}^n)$$

(and so does of course $\Xi^t = \operatorname{Op}(\langle \xi \rangle^t) = \langle D \rangle^t$). The special interest is that the "plus"/"minus" operators also define homeomorphisms related to $\overline{\mathbb{R}}_+^n$ and $\overline{\Omega}$, for all $s \in \mathbb{R}$:

$$(2.7) \quad \begin{aligned} \Xi_+^t : \dot{H}_p^s(\overline{\mathbb{R}}_+^n) &\xrightarrow{\sim} \dot{H}_p^{s-t}(\overline{\mathbb{R}}_+^n), & r^+ \Xi_-^t e^+ : \overline{H}_p^s(\mathbb{R}_+^n) &\xrightarrow{\sim} \overline{H}_p^{s-t}(\mathbb{R}_+^n); \\ \Lambda_+^{(t)} : \dot{H}_p^s(\overline{\Omega}) &\xrightarrow{\sim} \dot{H}_p^{s-t}(\overline{\Omega}), & r^+ \Lambda_-^{(t)} e^+ : \overline{H}_p^s(\Omega) &\xrightarrow{\sim} \overline{H}_p^{s-t}(\Omega); \end{aligned}$$

here $r^+ \Xi_-^t e^+$ resp. $r^+ \Lambda_-^{(t)} e^+$ are suitably extended to large negative s (cf. Rem. 1.1 and Th. 1.3 in [G15]). The first line in (2.7) also holds with Ξ replaced by Λ .

One has moreover that the operators Ξ_+^t and $r^+ \Xi_-^t e^+$ identify with each other's adjoints over $\overline{\mathbb{R}}_+^n$, because of the support preserving properties; more precisely,

$$(2.8) \quad \Xi_+^t : \dot{H}_{p'}^{-s}(\overline{\mathbb{R}}_+^n) \rightarrow \dot{H}_{p'}^{-s}(\overline{\mathbb{R}}_+^n) \text{ and } r^+ \Xi_-^t e^+ : \overline{H}_p^s(\mathbb{R}_+^n) \rightarrow \overline{H}_p^{s-t}(\mathbb{R}_+^n) \text{ are adjoints,}$$

for all $s \in \mathbb{R}$. The same holds for the operators $\Lambda_+^t, r^+ \Lambda_-^t e^+$, and there is a similar statement for $\Lambda_+^{(t)}$ and $r^+ \Lambda_-^{(t)} e^+$ relative to the set Ω .

The following special μ -transmission spaces introduced by Hörmander [H65] (for $p = 2$), cf. [G15], are particularly adapted to μ -transmission operators P ; we here just use them for real $\mu > -1$:

$$(2.9) \quad \begin{aligned} H_p^{\mu(s)}(\overline{\mathbb{R}}_+^n) &= \Xi_+^{-\mu} e^+ \overline{H}_p^{s-\mu}(\mathbb{R}_+^n) = \Lambda_+^{-\mu} e^+ \overline{H}_p^{s-\mu}(\mathbb{R}_+^n), & s > \mu - 1/p', \\ H_p^{\mu(s)}(\overline{\Omega}) &= \Lambda_+^{(-\mu)} e^+ \overline{H}_p^{s-\mu}(\Omega), & s > \mu - 1/p'. \end{aligned}$$

In fact, $r^+ P$ (of order m) maps them into $\overline{H}_p^{s-m}(\mathbb{R}_+^n)$ resp. $\overline{H}_p^{s-m}(\Omega)$ (cf. [G15] Sections 1.3, 2, 4), and they represent, when P is elliptic, the solution space for the homogeneous Dirichlet problem (1.5) with $f \in \overline{H}^{s-m}(\mathbb{R}_+^n)$ resp. $\overline{H}^{s-m}(\Omega)$. Moreover, $r^+ P$ maps $\mathcal{E}_\mu(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$, and $\mathcal{E}_\mu(\overline{\Omega})$ is the solution space for the Dirichlet problem with data in $C^\infty(\overline{\Omega})$. $\mathcal{E}_\mu(\overline{\Omega})$ is dense in $H_p^{\mu(s)}(\overline{\Omega})$ for all s , and $\bigcap_s H_p^{\mu(s)}(\overline{\Omega}) = \mathcal{E}_\mu(\overline{\Omega})$. (For $\Omega = \mathbb{R}_+^n$, $\mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ is dense in $H_p^{\mu(s)}(\overline{\mathbb{R}}_+^n)$ for all s .)

One has that $H_p^{\mu(s)}(\overline{\Omega}) \supset \dot{H}_p^s(\overline{\Omega})$, and the distributions are locally in H_p^s on Ω , but at the boundary they in general have a singular behavior (cf. [G15] Th. 5.4):

$$(2.10) \quad H_p^{\mu(s)}(\overline{\Omega}) \begin{cases} = \dot{H}_p^s(\overline{\Omega}) & \text{if } s \in]\mu - 1/p', \mu + 1/p[, \\ \subset e^+ d^\mu \overline{H}_p^{s-\mu}(\Omega) + \dot{H}_p^s(\overline{\Omega}) & \text{if } s > \mu + 1/p. \end{cases}$$

In the present paper, we use them with $\mu = a$ and $\mu = a - 1$. Note in particular that the spaces $H_p^{(a-1)(s)}(\overline{\Omega})$ satisfy

$$(2.11) \quad H_p^{(a-1)(s)}(\overline{\Omega}) \begin{cases} = \dot{H}_p^s(\overline{\Omega}) & \text{if } s \in]a-1-1/p', a-1/p'[, \\ \subset e^+ d^{a-1} \overline{H}_p^{s-a+1}(\Omega) + \dot{H}_p^s(\overline{\Omega}) & \text{if } s > a-1/p'. \end{cases}$$

When $0 < a < 1$, the factor d^{a-1} blows up at the boundary, so functions in these spaces can be viewed as “large”. However, it is worth remarking that they are L_p -integrable for $p' > 1/a$, $s \geq 0$. For $0 \leq s < a-1/p'$ this follows since $H_p^{(a-1)(s)}(\overline{\Omega}) = \dot{H}_p^s(\overline{\Omega})$ then, and for larger s , it follows since $H_p^{(a-1)(s')}(\overline{\Omega}) \subset H_p^{(a-1)(s)}(\overline{\Omega})$ when $s' > s$.

The following trace operators γ_k^μ (denoted $\gamma_{\mu,k}$ in [G14, G15, G16]), are defined on $\mathcal{E}_\mu(\overline{\Omega})$, and on $H_p^{\mu(s)}(\overline{\Omega})$ for s large enough in comparison with k :

$$(2.12) \quad \gamma_k^\mu u = \Gamma(\mu + 1 + k) \gamma_k(u/d^\mu),$$

where the γ_k are the standard trace operators $\gamma_k u = (\partial_n^k u)|_{\partial\Omega}$, $k = 0, 1, \dots$.

Our Green's formula will be shown for $p = 2$, since it is closely connected with L_2 -scalar products. The calculations of solution operators will be performed with $p = 2$, and supplied with remarks on how the results extend to H_p^s - and other spaces as in [G14, G15].

Further prerequisites are collected in the Appendix.

3. Green's formula in the simplest case.

We begin by an elementary explanation of the Dirichlet and Neumann boundary values on \mathbb{R}_+^n . Consider $u, v \in \mathcal{E}_{a-1}(\overline{\mathbb{R}_+^n}) \cap e^+ \mathcal{S}(\overline{\mathbb{R}_+^n})$, with $a > 0$. The boundary values $\gamma_k^{a-1} u = u_k$ are defined from the expansion

$$(3.1) \quad u(x) = u_0(x') I^{a-1}(x_n) + u_1(x') I^a(x_n) + \dots + u_k(x') I^{a-1+k}(x_n) + O(x_n^{a+k}),$$

where (as in [G15]) $I^\mu(x_n) = H(x_n) x_n^\mu / \Gamma(\mu + 1)$ when $\text{Re } \mu > -1$, H is the Heaviside function. The Gamma factor serves to normalize I^μ so that $\partial_{x_n} I^\mu = I^{\mu-1}$; this formula is also used to define the distribution for lower $\text{Re } \mu$. The expansion (3.1) follows from a Taylor expansion of $w(x) = u(x)/x_n^{a-1}$ for $x_n \searrow 0$:

$$(3.2) \quad u(x) = x_n^{a-1} w(x', 0) + x_n^a \partial_n w(x', 0) + \dots + x_n^{a+k-1} \frac{1}{k!} \partial_n^k w(x', 0) + O(x_n^{a+k}).$$

Note in particular that

$$(3.3) \quad \begin{aligned} \gamma_0^{a-1} u &= u_0 = \Gamma(a) \gamma_0 w = \Gamma(a) \gamma_0(u(x)/x_n^{a-1}), \\ \gamma_1^{a-1} u &= u_1 = \Gamma(a+1) \gamma_1 w = \Gamma(a+1) \gamma_1(u(x)/x_n^{a-1}); \end{aligned}$$

they are viewed as the *Dirichlet resp. Neumann traces* of $u \in \mathcal{E}_{a-1}(\overline{\mathbb{R}_+^n})$.

Besides the expansion (3.1) it will be convenient to use an expansion where the partially Fourier transformed terms have a factor $e^{-\sigma x_n}$, $\sigma = \langle \xi' \rangle$:

$$(3.4) \quad \begin{aligned} \mathcal{F}_{x' \rightarrow \xi'} u &= \acute{u}(\xi', x_n) = \hat{\varphi}_0(\xi') I^{a-1}(x_n) e^{-\sigma x_n} + \acute{u}'(\xi', x_n) \\ &= \hat{\varphi}_0(\xi') I^{a-1}(x_n) e^{-\sigma x_n} + \hat{\varphi}_1(\xi') I^a(x_n) e^{-\sigma x_n} + \acute{u}''(\xi', x_n) \\ &\sim \sum_{k \geq 0} \hat{\varphi}_k(\xi') I^{a-1+k}(x_n) e^{-\sigma x_n}; \text{ here} \\ \mathcal{F} u &\sim \sum_{k \geq 0} \hat{\varphi}_k(\xi') (\sigma + i \xi_n)^{-a-k}, \end{aligned}$$

using the formula

$$(3.5) \quad \mathcal{F}_{x_n \rightarrow \xi_n} [I^\mu(x_n) e^{-\sigma x_n}] = (\sigma + i\xi_n)^{-\mu-1}.$$

Correspondingly,

$$(3.6) \quad u = U_0 + u' = U_0 + U_1 + u'' \sim \sum_{k \geq 0} U_k, \text{ with } U_k = \mathcal{F}_{\xi' \rightarrow x'}^{-1} [\hat{\varphi}_k(\xi') I^{a-1+k}(x_n) e^{-\sigma x_n}].$$

Here we note that $u' \in \mathcal{E}_a$ and $u'' \in \mathcal{E}_{a+1}$, with $U_0 \in \mathcal{E}_{a-1}$, $U_1 \in \mathcal{E}_a$. So $\gamma_0^{a-1} u' = 0$ with $\gamma_0^{a-1} u = \gamma_0^{a-1} U_0$, and $\gamma_0^a u'' = 0$ with $\gamma_0^a u' = \gamma_0^a U_1$. (They are all in $e^+ \mathcal{S}(\overline{\mathbb{R}}_+^n)$.)

The transition between the coefficient sets $\{u_k\}$ and $\{\varphi_k\}$ can be found by comparison of the expansion of $\mathcal{F}_{x' \rightarrow \xi'} w = \dot{w}(\xi', x_n)$ with the expansion we get from $\dot{w}_e(\xi', x_n) = e^{\sigma x_n} \dot{w}(\xi', x_n) = e^{\sigma x_n} \dot{u}(\xi', x_n) / x_n^{a-1}$. Let us just do this in detail for the first two coefficients (sufficient for the present paper):

$$\begin{aligned} e^{\sigma x_n} \dot{u} &= x_n^{a-1} \dot{w}_e(\xi', 0) + x_n^a \partial_n \dot{w}_e(\xi', 0) + O(x_n^{a+1}) \\ &= e^{\sigma x_n} [x_n^{a-1} \hat{w}_0(\xi') + x_n^a (\hat{w}_1(\xi') + \sigma \hat{w}_0(\xi')) + O(x_n^{a+1})]. \end{aligned}$$

We see that $\hat{\varphi}_0 = \Gamma(a) \hat{w}_0 = \hat{u}_0$ and $\hat{\varphi}_1 = \Gamma(a+1) (\hat{w}_1 + \sigma \hat{w}_0) = \hat{u}_1 + a\sigma \hat{u}_0$, so

$$(3.7) \quad \varphi_0 = u_0, \quad \varphi_1 = u_1 + a \langle D' \rangle u_0,$$

where $\langle D' \rangle = \text{OP}(\langle \xi' \rangle)$. This allows us to relate the functions u_0, u_1 to boundary values of $\Xi_+^{a-1} u$. In view of (3.4), $\Xi_+^{a-1} = \text{OP}((\sigma + i\xi_n)^{a-1})$ has the effect

$$\Xi_+^{a-1} u \sim \mathcal{F}^{-1} \sum_{k \geq 0} \hat{\varphi}_k(\xi') (\sigma + i\xi_n)^{-1-k}.$$

Then since $\gamma_0 I^k = 0$ for $k = 1, 2, \dots$ (recall that the trace γ_0 is taken from \mathbb{R}_+^n),

$$\begin{aligned} \gamma_0 \Xi_+^{a-1} u &= \gamma_0 \mathcal{F}^{-1} (\hat{\varphi}_0(\xi') (\sigma + i\xi_n)^{-1} + \hat{\varphi}_1(\xi') (\sigma + i\xi_n)^{-2} + \dots) \\ &= \gamma_0 \mathcal{F}_{\xi' \rightarrow x'}^{-1} (\hat{\varphi}_0(\xi') I^0 e^{-\sigma x_n} + \hat{\varphi}_1(\xi') I^1 e^{-\sigma x_n} + \dots) \\ &= \mathcal{F}_{\xi' \rightarrow x'}^{-1} \hat{\varphi}_0(\xi') = \varphi_0 = u_0, \\ \gamma_0 \partial_n \Xi_+^{a-1} u &= \gamma_0 \partial_n \mathcal{F}_{\xi' \rightarrow x'}^{-1} (\hat{\varphi}_0(\xi') I^0 e^{-\sigma x_n} + \hat{\varphi}_1(\xi') I^1 e^{-\sigma x_n} + \dots) \\ &= \mathcal{F}_{\xi' \rightarrow x'}^{-1} (-\hat{\varphi}_0(\xi') \sigma + \hat{\varphi}_1(\xi')) = -\langle D' \rangle \varphi_0 + \varphi_1 = u_1 + (a-1) \langle D' \rangle u_0, \end{aligned}$$

cf. (3.7), showing that

$$(3.8) \quad u_0 = \gamma_0 \Xi_+^{a-1} u, \quad u_1 = \gamma_0 \partial_n \Xi_+^{a-1} u - (a-1) \langle D' \rangle u_0.$$

(It is used that $\mathcal{F}_{\xi' \rightarrow x'}^{-1} (\hat{\varphi}_0(\xi') I^{-1} e^{-\sigma x_n}) = \varphi_0(x') \delta(x_n)$ does not contribute to the boundary value from \mathbb{R}_+^n .) (3.8) was also shown in [G15] Sect. 5; related calculations occur in [G14], Appendix.

For $u' = u - U_0$ (cf. (3.6)), we note that since $u' \in \mathcal{E}_a$ with the expansion $U_1 + u''$, $u'' \in \mathcal{E}_{a+1}$,

$$(3.9) \quad \gamma_0^a u' = \varphi_1 = \gamma_1^{a-1} u + a \langle D' \rangle \gamma_0^{a-1} u = u_1 + a \langle D' \rangle u_0.$$

In particular, u' itself satisfies, since $\gamma_0^{a-1} u' = 0$ and $\gamma_1^{a-1} u' = \gamma_1^{a-1} u$,

$$(3.10) \quad \gamma_0^a u' = \gamma_1^{a-1} u'.$$

In other words: *When $u \in \mathcal{E}_{a-1}$ is such that the Dirichlet trace $\gamma_0^{a-1} u$ of u vanishes, then the Neumann trace equals $\gamma_0^a u$.*

There are similar formulas for v , expanded as in (3.6) with coefficients ψ_k :

$$(3.11) \quad v = V_0 + v' = V_0 + V_1 + v'', \text{ with } V_k = \mathcal{F}_{\xi' \rightarrow x'}^{-1} [\hat{\psi}_k(\xi') I^{a-1+k} e^{-\sigma x_n}].$$

The formula (3.8) allows us to deduce mapping properties of the γ_k^{a-1} in Sobolev spaces. Recall from Section 2 that $\mathcal{E}_{a-1}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ is dense in $H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{1-a} e^+ \overline{H}^{s-a+1}(\mathbb{R}_+^n)$. Then since $\gamma_k: \overline{H}^{s-a+1}(\mathbb{R}_+^n) \rightarrow H^{s-a-k+\frac{1}{2}}(\mathbb{R}^{n-1})$, the Dirichlet and Neumann traces extend by continuity to continuous operators:

$$(3.12) \quad \begin{aligned} \gamma_0^{a-1}: H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n) &\rightarrow H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1}), \quad s > a - \frac{1}{2}, \\ \gamma_1^{a-1}: H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n) &\rightarrow H^{s-a-\frac{1}{2}}(\mathbb{R}^{n-1}), \quad s > a + \frac{1}{2}. \end{aligned}$$

In this context, we note that (since $\varphi_0 = u_0$)

$$\begin{aligned} U_0 &= \mathcal{F}_{\xi' \rightarrow x'}^{-1} [\hat{u}_0(\xi') I^{a-1}(x_n) e^{-\sigma x_n}] = \mathcal{F}_{\xi \rightarrow x}^{-1} [\hat{u}_0(\xi') (\sigma + i\xi_n)^{-a}] \\ &= \Xi_+^{1-a} \mathcal{F}_{\xi \rightarrow x}^{-1} [\hat{u}_0(\xi') (\sigma + i\xi_n)^{-1}] = \Xi_+^{1-a} e^+ K_0 u_0, \end{aligned}$$

where K_0 is the well-known Poisson operator $K_0 \varphi = \mathcal{F}_{\xi' \rightarrow x'}^{-1} [\hat{\varphi}(\xi') r^+ e^{-\langle \xi' \rangle x_n}]$, defining a right-inverse of γ_0 ; its symbol is $(\sigma + i\xi_n)^{-1} = \chi_+^{-1}$. It maps $H^t(\mathbb{R}^{n-1}) \rightarrow \overline{H}^{t+\frac{1}{2}}(\mathbb{R}_+^n)$ for all $t \in \mathbb{R}$, so

$$K_0^{a-1} \equiv \Xi_+^{1-a} e^+ K_0: H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1}) \rightarrow H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n), \quad s > a - \frac{1}{2},$$

defining a right-inverse of γ_0^{a-1} . (This was also shown in [G15] Cor. 5.3.)

The various identities shown above, and the remark after (3.10), extend to these spaces, for s suitably chosen.

We shall now give a relatively elementary proof of the desired Green's formula for $P = (1 - \Delta)^a$ on \mathbb{R}_+^n . Since $(1 + |\xi|^2)^a = (\langle \xi' \rangle - i\xi_n)^a (\langle \xi' \rangle + i\xi_n)^a$, $(1 - \Delta)^a = \Xi_-^a \Xi_+^a$. In view of (3.5):

$$(3.13) \quad \begin{aligned} \Xi_+^a U_k &= \mathcal{F}^{-1} [(\sigma + i\xi_n)^a \hat{\varphi}_k(\xi') (\sigma + i\xi_n)^{-a-k}] \\ &= \mathcal{F}_{\xi' \rightarrow x'}^{-1} [\hat{\varphi}_k(\xi') I^{k-1} e^{-\sigma x_n}] \text{ for } k \in \mathbb{N}_0, \text{ in particular,} \\ \Xi_+^a U_0 &= \mathcal{F}^{-1} [(\sigma + i\xi_n)^a \hat{\varphi}_0(\xi') (\sigma + i\xi_n)^{-a}] = \varphi_0(x') \otimes \delta(x_n), \\ \Xi_+^a U_1 &= \mathcal{F}^{-1} [(\sigma + i\xi_n)^a \hat{\varphi}_1(\xi') (\sigma + i\xi_n)^{-a-1}] = \mathcal{F}_{\xi' \rightarrow x'}^{-1} [\hat{\varphi}_1(\xi') H e^{-\sigma x_n}]. \end{aligned}$$

An application of Ξ_-^a gives

$$\begin{aligned} (1 - \Delta)^a U_0 &= \Xi_-^a(\varphi_0(x') \otimes \delta(x_n)), \text{ supported in } \overline{\mathbb{R}}_-^n, \\ (1 - \Delta)^a U_1 &= \mathcal{F}^{-1}[(\sigma - i\xi_n)^a \hat{\varphi}_1(\xi')(\sigma + i\xi_n)^{-1}]. \end{aligned}$$

From the first line we conclude for $P = (1 - \Delta)^a$:

$$(3.14) \quad r^+ P U_0 = 0, \text{ hence } r^+ P u = r^+ P u'.$$

Now $\int_{\mathbb{R}_+^n} r^+ P u \bar{v} dx$ will be worked out. We know from [G15] Th. 4.2 that $r^+ P$ maps $H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n)$ into $\overline{H}^{s-2a}(\mathbb{R}_+^n)$ (and $\mathcal{E}_{a-1}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ into $\bigcap_t \overline{H}^t(\mathbb{R}_+^n)$). For large s , $\overline{H}^{s-2a}(\mathbb{R}_+^n)$ is a space of continuous functions, and $H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n)$ is as such, supplied with continuous functions multiplied by x_n^{a-1} , so $r^+ P u \bar{v}$ is integrable for $x_n \rightarrow 0$. For smaller s , we need an interpretation as a duality.

Note first (cf. (3.11) and (3.14)) that

$$(3.15) \quad \int_{\mathbb{R}_+^n} r^+ P u \bar{v} dx = \int_{\mathbb{R}_+^n} r^+ P u' \bar{v} dx = \int_{\mathbb{R}_+^n} r^+ P u' \bar{v}' dx + \int_{\mathbb{R}_+^n} r^+ P u' \bar{V}_0 dx.$$

In the term $\int_{\mathbb{R}_+^n} r^+ P u' \bar{v}' dx$, $v' \in x_n^a \overline{H}^{s-a}(\mathbb{R}_+^n) + \dot{H}^s(\overline{\mathbb{R}}_+^n)$ does not give integrability problems. This integral will be left unchanged, to match a similar integral with P applied to v' . It is the last integral that will be reduced to an integral of boundary values, and which we now study more closely.

From now on, take u and v in $H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{1-a} e^+ \overline{H}^{s-a+1}(\mathbb{R}_+^n)$. The following calculations are very similar to those in the proof of [G16], Th. 3.1. Let $s > a + \frac{1}{2}$, then (for small $\varepsilon > 0$)

$$(3.16) \quad \begin{aligned} u, v, U_0, V_0 &\in H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{1-a} e^+ \overline{H}^{s-a+1}(\mathbb{R}_+^n) \subset \Xi_+^{1-a} \dot{H}^{\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n) = \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \\ u', v' &\in H^{a(s)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-a} e^+ \overline{H}^{s-a}(\mathbb{R}_+^n) \subset \Xi_+^{-a} \dot{H}^{\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n) = \dot{H}^{a+\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \\ r^+ P u, r^+ P u', r^+ P v, r^+ P v' &\in \overline{H}^{s-2a}(\mathbb{R}_+^n) \subset \overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \\ u_k, v_k &\in H^{s-a-k+\frac{1}{2}}(\mathbb{R}^{n-1}) \subset H^{1-k+\varepsilon}(\mathbb{R}^{n-1}), \quad k = 0, 1. \end{aligned}$$

We then find that $I = \int_{\mathbb{R}_+^n} r^+ P u' \bar{V}_0 dx$ can be interpreted in these larger spaces as

$$(3.17) \quad I \equiv \langle r^+ P u', V_0 \rangle_{\overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)}$$

(sesquilinear duality). Since $u' \in H^{a(s)}(\overline{\mathbb{R}}_+^n)$, $r^+ P u' = r^+ \Xi_-^a e^+ r^+ \Xi_+^a u' = r^+ \Xi_-^a e^+ w$, where $w = r^+ \Xi_+^a u' \in \overline{H}^{s-a}(\mathbb{R}_+^n)$; here $r^+ \Xi_-^a e^+$ maps $\overline{H}^t(\mathbb{R}_+^n)$ to $\overline{H}^{t-a}(\mathbb{R}_+^n)$ for all $t \in \mathbb{R}$, with adjoint $\Xi_+^a: \dot{H}^{a-t}(\overline{\mathbb{R}}_+^n) \rightarrow \dot{H}^{-t}(\overline{\mathbb{R}}_+^n)$, cf. (2.7), (2.8). Therefore, by (3.13),

$$(3.18) \quad \begin{aligned} I &= \langle r^+ \Xi_-^a e^+ r^+ \Xi_+^a u', V_0 \rangle_{\overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} \\ &= \langle w, \Xi_+^a V_0 \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} = \langle w, \psi_0(x') \otimes \delta(x_n) \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)}. \end{aligned}$$

Recall moreover from distribution theory (cf. e.g. [G09] p. 307) that the “two-sided” trace operator $\tilde{\gamma}_0: v(x) \mapsto \tilde{\gamma}_0 v = v(x', 0)$ has the mapping $\tilde{\gamma}_0^*: \varphi(x') \mapsto \varphi(x') \otimes \delta(x_n)$ as adjoint, with continuity properties

$$\tilde{\gamma}_0: H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^n) \rightarrow H^\varepsilon(\mathbb{R}^{n-1}), \quad \tilde{\gamma}_0^*: H^{-\varepsilon}(\mathbb{R}^{n-1}) \rightarrow H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^n), \text{ for } \varepsilon > 0.$$

Here $\tilde{\gamma}_0^* \varphi$ is supported in $\{x_n = 0\}$, hence lies in $\dot{H}^{-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)$. Since $w \in \overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n)$, it has an extension $W \in H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^n)$ with $w = r^+ W$, and $\gamma_0 w = \tilde{\gamma}_0 W$. Then

$$\begin{aligned} I &= \langle w, \psi_0(x') \otimes \delta(x_n) \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} = \langle W, \psi_0(x') \otimes \delta(x_n) \rangle_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^n), H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^n)} \\ &= \langle W, \tilde{\gamma}_0^* \psi_0 \rangle_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^n), H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^n)} = \langle \tilde{\gamma}_0 W, \psi_0 \rangle_{H^\varepsilon(\mathbb{R}^{n-1}), H^{-\varepsilon}(\mathbb{R}^{n-1})} = (\gamma_0 w, v_0)_{L_2(\mathbb{R}^{n-1})}, \end{aligned}$$

since $\psi_0 = v_0 \in L_2(\mathbb{R}^{n-1})$. Finally, since $u' = U_1 + u''$ with $\gamma_0 w = \gamma_0 \Xi_+^a U_1 = \varphi_1 = u_1 + a\langle D' \rangle u_0$ by (3.6), (3.7), (3.13), we conclude:

Lemma 3.1. *Let $P = (1 - \Delta)^a$, and let $u, v \in H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n)$, $s > a + \frac{1}{2}$, with Dirichlet and Neumann boundary values $\gamma_0^{a-1} u = u_0$ and $\gamma_1^{a-1} u = u_1$ (and similarly for v), as defined above. Let $V_0 = \mathcal{F}_{\xi' \rightarrow x'}^{-1}[\hat{v}_0(\xi') I^{a-1}(x_n) e^{-\langle \xi' \rangle x_n}]$. Then $\int_{\mathbb{R}_+^n} r^+ P u \bar{V}_0 dx$, understood as the duality (3.17), satisfies*

$$\begin{aligned} (3.19) \quad \langle r^+ P u, V_0 \rangle_{\overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} &= \langle r^+ P u', V_0 \rangle_{\overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} \\ &= (u_1 + a\langle D' \rangle u_0, v_0)_{L_2(\mathbb{R}^{n-1})}. \end{aligned}$$

From this we obtain the Green's formula:

Theorem 3.2. *Let $P = (1 - \Delta)^a$, and let $u, v \in H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n)$ with $s > a + \frac{1}{2}$. Then*

$$\begin{aligned} (3.20) \quad \langle r^+ P u, v \rangle_{\overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} &- \langle u, r^+ P v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n)} \\ &= \int_{\mathbb{R}^{n-1}} (\gamma_1^{a-1} u \gamma_0^{a-1} \bar{v} - \gamma_0^{a-1} u \gamma_1^{a-1} \bar{v}) dx'. \end{aligned}$$

Here when $s \geq 2a$, the left-hand side can be written as an ordinary integral

$$(3.21) \quad \int_{\mathbb{R}_+^n} (r^+ P u \bar{v} dx - u \overline{r^+ P v}) dx.$$

Proof. It follows from (3.15) and Lemma 3.1 together that when $s > a + \frac{1}{2}$,

$$\begin{aligned} (3.22) \quad \langle r^+ P u, v \rangle_{\overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} &= \langle r^+ P u', v' \rangle_{\overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} + (u_1 + a\langle D' \rangle u_0, v_0)_{L_2(\mathbb{R}^{n-1})}. \end{aligned}$$

There is a similar formula obtained by interchanging u and v and conjugating:

$$\begin{aligned} (3.23) \quad \langle u, r^+ P v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n)} &= \langle u', r^+ P v' \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n)} + (u_0, v_1 + a\langle D' \rangle v_0)_{L_2(\mathbb{R}^{n-1})}. \end{aligned}$$

By Th. 4.1 of [G16],

$$\langle r^+ Pu', v' \rangle_{\overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} - \langle u', r^+ Pv' \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n)} = 0.$$

The dualities written there are consistent with the present ones, since $u', v' \in H^{a(s)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-a} e^+ \overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n) \subset \dot{H}^{1-\varepsilon}$ and $Pu', Pv' \in \overline{H}^{s-2a}(\mathbb{R}_+^n) \subset \overline{H}^{\frac{1}{2}-a+\varepsilon}(\mathbb{R}_+^n)$.

Formula (3.20) then follows by taking the difference of (3.22) and (3.23), using that $a\langle D' \rangle$ is selfadjoint.

If $s \geq 2a$, then $r^+ Pu, r^+ Pv \in \overline{H}^{s-2a}(\mathbb{R}_+^n) \subset L_2(\mathbb{R}_+^n)$. Moreover, $u, v \in x_n^{a-1} \overline{H}^{a+1}(\mathbb{R}_+^n) + \dot{H}^{2a}(\overline{\mathbb{R}}_+^n)$, cf. (2.11). So $r^+ Pu \bar{v}$ and $u \overline{r^+ Pv}$ are functions, and we can write the dualities as in (3.21), keeping the interpretation in mind. \square

4. Green's formula for variable-coefficient operators.

Let $P = \text{OP}(p(x, \xi))$ be a classical ψ do on \mathbb{R}^n of order $2a > 0$ (global estimates), with symbol $p(x, \xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi)$. We assume that p is *even*, i.e.,

$$p_j(x, -\xi) = (-1)^j p_j(x, \xi) \text{ for } |\xi| \geq 1, \text{ all } x.$$

Let Ω be a smooth bounded subset of \mathbb{R}^n , or $\Omega = \mathbb{R}_+^n$. The evenness implies that p satisfies (2.4) with $m = 2a$, $\mu = a$, so p (or P) has the a -transmission property at Ω .

Green's formula for such operators will first be shown in the case $\Omega = \mathbb{R}_+^n$, and afterwards generalized to the curved case. The main strategy is to reduce as much as possible to rules from the Boutet de Monvel calculus, where the issues of operators passing to and from the boundary are dealt with in a systematic way (in the 0-transmission case). We refer the reader to e.g. [G09] for a general presentation of the calculus; a few important ingredients are collected in the Appendix here.

Theorem 4.1. *Let P be a classical ψ do on \mathbb{R}^n of order $2a > 0$ with even symbol. The following Green's formula holds for $u, v \in H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n)$ when $s > a + \frac{1}{2}$:*

$$(4.1) \quad \begin{aligned} & \langle r^+ Pu, v \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)} - \langle u, r^+ P^* v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \overline{H}^{-a+\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n)} \\ &= (s_0 \gamma_1^{a-1} u, \gamma_0^{a-1} v) - (s_0 \gamma_0^{a-1} u, \gamma_1^{a-1} v) + (B \gamma_0^{a-1} u, \gamma_0^{a-1} v), \end{aligned}$$

with $L_2(\mathbb{R}^{n-1})$ -scalar products in the right-hand side; here $s_0 = p_0(x', 0, 0, 1)$, and B is a first-order ψ do on \mathbb{R}^{n-1} whose symbol equals the jump at $z_n = 0$ in the bounded part of $\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} q(x', 0, \xi)$, where q is the symbol of $\Xi_-^{-a} P \Xi_+^{-a}$. The right-hand side can also be written

$$(4.2) \quad \begin{aligned} & \Gamma(a) \Gamma(a+1) \int_{\mathbb{R}^{n-1}} \left(s_0 \gamma_1 \left(\frac{u}{x_n^{a-1}} \right) \gamma_0 \left(\frac{\bar{v}}{x_n^{a-1}} \right) - s_0 \gamma_0 \left(\frac{u}{x_n^{a-1}} \right) \gamma_1 \left(\frac{\bar{v}}{x_n^{a-1}} \right) + \right. \\ & \quad \left. + a B \gamma_0 \left(\frac{u}{x_n^{a-1}} \right) \gamma_0 \left(\frac{\bar{v}}{x_n^{a-1}} \right) \right) dx'. \end{aligned}$$

When $s \geq 2a$, the dualities in the left-hand side can be written as integrals over Ω .

Proof. Define $Q = \Xi_-^{-a} P \Xi_+^{-a}$, it is a generalized ψ do of order 0, with a symbol $q(x, \xi)$ that is the sum of $s_0(x) = p_0(x, 0, 1)$ and a function in $S^0(\mathcal{H}_{-1})$ (notation explained in the

Appendix). The principal symbol $q_0(x, \xi)$ is the ψ do symbol $p_0(x, \xi)[\xi]^{-2a}$. We also need the (x', y_n) -form $q'(x', y_n, \xi)$ related to q by (A.11). Now we can write P as $P = \Xi_-^a Q \Xi_+^a$.

(In the following study, Ξ_\pm^t and Λ_\pm^t can be used equally well. With the use of Λ_\pm^t and the corresponding choice of Q , the calculations stay within true pseudodifferential operators as much as possible. With Ξ_\pm^t the formulas are simpler and more direct; here we draw on the fact that Q has symbol in $S^0(\mathcal{H}_{-1})$ plus smooth functions, where the rules for Poisson and trace operators we need are still valid.)

Using the description of γ_0^{a-1} and K_0^{a-1} given in Section 3, we decompose a function $u \in H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n)$ as

$$(4.3) \quad u = u' + K_0^{a-1} \gamma_0^{a-1} u; \text{ here } u' \in H^{a(s)}(\overline{\mathbb{R}}_+^n) \text{ since } \gamma_0^{a-1} u' = 0.$$

There is a similar decomposition for v , and we denote

$$(4.4) \quad \gamma_0^{a-1} u = u_0, \quad \gamma_0^{a-1} v = v_0, \text{ both in } H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1}).$$

We assume $s > a + \frac{1}{2}$, so instead of s we can insert $a + \frac{1}{2} + \varepsilon$, and

$$(4.5) \quad \begin{aligned} u, v, K_0^{a-1} \gamma_0^{a-1} u \text{ and } K_0^{a-1} \gamma_0^{a-1} v &\in H^{(a-1)(a+\frac{1}{2}+\varepsilon)}(\overline{\mathbb{R}}_+^n) \subset \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \\ u_0, v_0 &\in H^{1+\varepsilon}(\mathbb{R}^{n-1}), \\ u', v' &\in H^{a(a+\frac{1}{2}+\varepsilon)}(\overline{\mathbb{R}}_+^n) \subset \dot{H}^{a+\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \\ r^+ P u, r^+ P^* v &\in \overline{H}^{-a+\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n). \end{aligned}$$

Then $\langle r^+ P u, v \rangle$ can be interpreted as

$$(4.6) \quad \langle r^+ P u, v \rangle = \langle r^+ P u, v \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}, \dot{H}^{a-\frac{1}{2}-\varepsilon}}.$$

This expression is split into four parts by applying (4.3) to u and v :

$$(4.7) \quad \begin{aligned} \langle r^+ P u, v \rangle &= I_1 + I_2 + I_3 + I_4, \\ I_1 &= \langle r^+ P u', v' \rangle, \quad I_2 = \langle r^+ P K_0^{a-1} u_0, v' \rangle, \\ I_3 &= \langle r^+ P u', K_0^{a-1} v_0 \rangle, \quad I_4 = \langle r^+ P K_0^{a-1} u_0, K_0^{a-1} v_0 \rangle. \end{aligned}$$

I_1 will be kept unchanged, to match a similar term with P^* later.

For I_2 we observe (for small ε), using (2.8):

$$(4.8) \quad \begin{aligned} I_2 &= \langle r^+ P K_0^{a-1} u_0, v' \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}, \dot{H}^{a-\frac{1}{2}-\varepsilon}} \\ &= \langle r^+ \Xi_-^a e^+ r^+ Q \Xi_+^a \Xi_+^{1-a} e^+ K_0 u_0, v' \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}, \dot{H}^{a-\frac{1}{2}-\varepsilon}} \\ &= \langle r^+ Q \Xi_+^a \Xi_+^{1-a} e^+ K_0 u_0, \Xi_+^a v' \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} \\ &= \langle r^+ Q \Xi_+^1 e^+ K_0 u_0, \Xi_+^a v' \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}}. \end{aligned}$$

By the rules of calculus for ψ dbo's, using the symbol q' of Q in (x', y_n) -form,

$$(4.9) \quad r^+ Q \Xi_+^1 e^+ K_0 = \text{OPK}(h^+(q' \# \chi_+^1 \chi_+^{-1})) = \text{OPK}(h^+ q'(x', 0, \xi)),$$

which leads to

$$(4.10) \quad I_2 = \langle \text{OPK}(h^+ q') u_0, \Xi_+^a v' \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}}.$$

If $Q = I$ we get zero here, but for general Q there is a nontrivial contribution from $h^+ q'$. Note that with $s_0(x) = p_0(x, 0, 1)$,

$$(4.11) \quad q(x, \xi) = s_0(x) + h_{-1} q(x, \xi), \quad h^+ q = h^+ h_{-1} q, \quad h^- q = s_0 + h_{-1}^- q,$$

with similar rules for $q'(x', y_n, \xi)$.

Now consider I_3 . Here, using (2.8) and the adjoint K_0^* of K_0 ,

$$(4.12) \quad \begin{aligned} I_3 &= \langle r^+ P u', K_0^{a-1} v_0 \rangle = \langle r^+ \Xi_-^a Q \Xi_+^a u', \Xi_+^{1-a} e^+ K_0 v_0 \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}, \dot{H}^{a-\frac{1}{2}-\varepsilon}} \\ &= \langle r^+ \Xi_-^{1-a} e^+ r^+ \Xi_-^a e^+ r^+ Q \Xi_+^a u', K_0 v_0 \rangle_{\overline{H}^{-\frac{1}{2}+\varepsilon}, \dot{H}^{\frac{1}{2}-\varepsilon}} \\ &= \langle r^+ \Xi_-^1 e^+ r^+ Q \Xi_+^a u', K_0 v_0 \rangle_{\overline{H}^{-\frac{1}{2}+\varepsilon}, \dot{H}^{\frac{1}{2}-\varepsilon}} \\ &= \langle K_0^* r^+ \Xi_-^1 Q \Xi_+^a u', v_0 \rangle_{H^\varepsilon(\mathbb{R}^{n-1}), H^{-\varepsilon}(\mathbb{R}^{n-1})} = (K_0^* r^+ \Xi_-^1 Q \Xi_+^a u', v_0)_{L_2(\mathbb{R}^{n-1})}, \end{aligned}$$

since v_0 is in $L_2(\mathbb{R}^{n-1})$. It is used that \overline{H}^t identifies with \dot{H}^t for $|t| < \frac{1}{2}$ (there the indication e^+ is understood).

Denote $\Xi_+^a u' = w \in e^+ \overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n)$. Observe that

$$K_0^* r^+ \Xi_-^1 Q w = \text{OPT}(h^-(\chi_-^{-1} \chi_-^1 \# q)) w = \text{OPT}(h^- q(x', 0, \xi)) w,$$

by the rules of calculus, so

$$(4.13) \quad \begin{aligned} I_3 &= (\text{OPT}(h^- q) \Xi_+^a u', v_0)_{L_2(\mathbb{R}^{n-1})} = ((s_0 \gamma_0 + \text{OPT}(h_{-1}^- q)) \Xi_+^a u', v_0)_{L_2(\mathbb{R}^{n-1})} \\ &= (s_0 \gamma_0^a u', v_0)_{L_2(\mathbb{R}^{n-1})} + (\text{OPT}(h_{-1}^- q) \Xi_+^a u', v_0)_{L_2(\mathbb{R}^{n-1})}. \end{aligned}$$

The first term is expected from Theorem 3.2, and there is a nontrivial extra term.

Finally, consider I_4 : Here

$$(4.14) \quad \begin{aligned} I_4 &= \langle r^+ P K_0^{a-1} u_0, K_0^{a-1} v_0 \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}, \dot{H}^{a-\frac{1}{2}-\varepsilon}} \\ &= \langle r^+ \Xi_-^a Q \Xi_+^a \Xi_+^{1-a} e^+ K_0 u_0, \Xi_+^{1-a} e^+ K_0 v_0 \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}, \dot{H}^{a-\frac{1}{2}-\varepsilon}} \\ &= \langle r^+ \Xi_-^1 e^+ r^+ Q \Xi_+^1 e^+ K_0 u_0, K_0 v_0 \rangle_{\overline{H}^{-\frac{1}{2}+\varepsilon}, \dot{H}^{\frac{1}{2}-\varepsilon}} \\ &= \langle K_0^* r^+ \Xi_-^1 Q \Xi_+^1 e^+ K_0 u_0, v_0 \rangle_{H^\varepsilon(\mathbb{R}^{n-1}), H^{-\varepsilon}(\mathbb{R}^{n-1})} \\ &= (\mathcal{B} u_0, v_0)_{L_2(\mathbb{R}^{n-1})}, \end{aligned}$$

where $\mathcal{B} = K_0^* r^+ \Xi_-^1 e^+ r^+ Q \Xi_+^1 e^+ K_0$ is a certain ψ do on \mathbb{R}^{n-1} of order 1. As in (4.9), the symbol of the Poisson operator $r^+ Q \Xi_+^1 e^+ K_0$ is $h^+ q'(x', 0, \xi)$, which by composition with

$r^+\Xi_-^1 e^+$ to the left gives a Poisson operator with symbol $h^+(\chi_-^1 \# h^+ q')$; hence the symbol $\mathfrak{b}(x', \xi')$ of \mathcal{B} satisfies (cf. (A.14)ff.),

$$\begin{aligned}
(4.15) \quad \mathfrak{b}(x', \xi') &= \frac{1}{2\pi} \int^+ \chi_-^{-1} \# h^+(\chi_-^1 \# h^+ q'(x', 0, \xi)) d\xi_n \\
&= \frac{1}{2\pi} \int^+ \chi_-^{-1} \# (\chi_-^1 \# h^+ q'(x', 0, \xi) - h^-(\chi_-^1 \# h^+(q'(x', 0, \xi)))) d\xi_n \\
&= \frac{1}{2\pi} \int^+ (h^+ q'(x', 0, \xi) - \chi_-^{-1} \# h^-(\chi_-^1 \# h^+(q'(x', 0, \xi)))) d\xi_n \\
&= \frac{1}{2\pi} \int^+ h^+ q'(x', 0, \xi) d\xi_n;
\end{aligned}$$

we here used moreover that the plus-integral vanishes on \mathcal{H}^- . (The Leibniz product $\#$ pertains to the x' -dependence.) Now observe that with $\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} q'(x', 0, \xi)$ denoted $\check{q}'(x', 0, \xi', z_n)$,

$$\frac{1}{2\pi} \int^+ h^+ q'(x', 0, \xi) d\xi_n = \lim_{z_n \rightarrow 0+} \check{q}'(x', 0, \xi', z_n),$$

cf. e.g. [G09] Lemma 10.18. (It should be noted that since $q'(x', 0, \xi)$ is the sum of a function $f(x', \xi) \in S^0(\mathcal{H}_{-1})$ and the function $s_0(x', 0)$, $\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} q'(x', 0, \xi)$ is the sum of a function $\check{f} \in S^0(e^- \mathcal{S}(\overline{\mathbb{R}}_-) \oplus e^+ \mathcal{S}(\overline{\mathbb{R}}_+))$ and the distribution $s_0(x', 0) \delta(z_n)$, where the latter does not enter in the limit from the right.) Moreover, in view of the formula (A.11),

$$\check{q}'(x', 0, \xi', z_n) \sim \sum_{j \in \mathbb{N}_0} \frac{1}{j!} z_n^j \partial_{x_n}^j \check{q}(x', 0, \xi', z_n),$$

hence

$$\lim_{z_n \rightarrow 0+} \check{q}'(x', 0, \xi', z_n) = \lim_{z_n \rightarrow 0+} \check{q}(x', 0, \xi', z_n),$$

since the positive powers of z_n vanish at 0. It follows that

$$(4.16) \quad \mathfrak{b}(x', \xi') = \lim_{z_n \rightarrow 0+} \check{q}(x', 0, \xi', z_n) = \frac{1}{2\pi} \int^+ h^+ q(x', 0, \xi) d\xi_n.$$

There is a similar decomposition of $\langle u, r^+ P^* v \rangle$ in four terms I'_1, I'_2, I'_3, I'_4 . We here note that Q^* has the symbol in y -form $\bar{q}(y, \xi)$, where

$$(4.17) \quad \bar{q} = \bar{s}_0 + \overline{h_{-1}q}, \quad h^+ \bar{q} = h^+(\overline{h_{-1}q}) = \overline{h_{-1}q}, \quad h^- \bar{q} = \bar{s}_0 + h^-(\overline{h_{-1}q}) = \bar{s}_0 + \overline{h^+q}.$$

The symbol of Q^* in (y', x_n) -form is $\bar{q}'(y', x_n, \xi)$, satisfying similar rules.

We here find the formulas

$$\begin{aligned}
(4.18) \quad I'_1 &= \langle u', r^+ P^* v' \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}, \overline{H}^{-a+\frac{1}{2}+\varepsilon}}, \\
I'_2 &= \langle \Xi_+^a u', \text{OPK}(\overline{h_{-1}q(y', 0, \xi)}) v_0 \rangle_{\dot{H}^{-\frac{1}{2}-\varepsilon}, \overline{H}^{\frac{1}{2}+\varepsilon}}, \\
I'_3 &= \langle u_0, \bar{s}_0 \gamma_0^a v' \rangle + \langle v_0, \text{OPT}(\overline{h^+ q'(y', 0, \xi)}) \Xi_+^a v' \rangle, \\
I'_4 &= (u_0, \mathcal{B}' v_0)_{L_2(\mathbb{R}^{n-1})},
\end{aligned}$$

where the operators derived from q are in y' -form. The operator \mathcal{B}' is defined as

$$\mathcal{B}' = K_0^* r^+ \Xi_-^1 e^+ r^+ Q^* \Xi_+^1 e^+ K_0;$$

it has the symbol (reduced as in (4.15))

$$\begin{aligned} \mathfrak{b}'(y', \xi') &= \frac{1}{2\pi} \int^+ \chi_-^{-1} h^+ (\chi_-^1 h^+ \bar{q}(y', 0, \xi)) d\xi_n \\ &= \frac{1}{2\pi} \int^+ h^+ \bar{q}(y', 0, \xi) d\xi_n = \lim_{z_n \rightarrow 0^+} \mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \bar{q}(y', 0, \xi', z_n). \end{aligned}$$

In view of (4.17), $h^+ \bar{q} = \overline{h_{-1}^- q}$. We note that $h^+ q$ and $\overline{h_{-1}^- q}$ can be quite different, so \mathcal{B} and \mathcal{B}' (or its adjoint) are in general different from one another.

The adjoint of \mathcal{B}' is $\mathcal{B}'^* = \text{OP}'(\overline{\mathfrak{b}'}(x', \xi'))$. It is well-known (and easily checked, cf. e.g. [G09] p. 118) that for a distribution $\varphi(\xi_n)$, $\overline{[\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \overline{\varphi}]}(z_n) = [\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \varphi](-z_n)$. Then

$$\begin{aligned} I'_4 &= (\mathcal{B}'^* u_0, v_0), \text{ where } \mathcal{B}'^* = \text{OP}'(\overline{\mathfrak{b}'}(x', \xi')), \\ (4.19) \quad \overline{\mathfrak{b}'}(x', \xi') &= \lim_{z_n \rightarrow 0^+} \overline{\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \bar{q}(x', 0, \xi', z_n)} = \lim_{z_n \rightarrow 0^+} \check{q}(x', 0, \xi', -z_n) \\ &= \lim_{z_n \rightarrow 0^-} \check{q}(x', 0, \xi', z_n). \end{aligned}$$

The full right-hand side in (4.1) is $I_1 + I_2 + I_3 + I_4 - I'_1 - I'_2 - I'_3 - I'_4$, that we can now calculate. Here we find:

$$I_1 - I'_1 = 0,$$

by [G16] Th. 4.1. Next,

$$\begin{aligned} I_2 - I'_2 &= \langle \text{OPK}(h^+ q') u_0, \Xi_+^a v' \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} - \langle \Xi_+^a u', \text{OPK}(\overline{h_{-1}^- q}) v_0 \rangle_{\dot{H}^{-\frac{1}{2}-\varepsilon}, \overline{H}^{\frac{1}{2}+\varepsilon}} \\ &= \langle \text{OPK}(h^+ q') u_0, \Xi_+^a v' \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} - \langle \text{OPT}(h_{-1}^- q) \Xi_+^a u', v_0 \rangle_{H^{-\varepsilon}(\mathbb{R}^{n-1}), H^\varepsilon(\mathbb{R}^{n-1})}, \\ I_3 - I'_3 &= \langle s_0 \gamma_0^a u', v_0 \rangle + \langle \text{OPT}(h_{-1}^- q) \Xi_+^a u', v_0 \rangle - \langle v_0, \bar{s}_0 \gamma_0^a v' \rangle - \langle u_0, \text{OPT}(\overline{h^+ q'}) \Xi_+^a v' \rangle \\ &= \langle s_0 (\gamma_1^{a-1} u + a \langle D' \rangle u_0), v_0 \rangle - \langle s_0 u_0, \gamma_1^{a-1} v + a \langle D' \rangle v_0 \rangle \\ &\quad + \langle \text{OPT}(h_{-1}^- q) \Xi_+^a u', v_0 \rangle - \langle u_0, \text{OPT}(\overline{h^+ q'}) \Xi_+^a v' \rangle \\ &= \langle s_0 \gamma_1^{a-1} u, v_0 \rangle - \langle s_0 u_0, \gamma_1^{a-1} v \rangle \\ &\quad + \langle \text{OPT}(h_{-1}^- q) \Xi_+^a u', v_0 \rangle - \langle \text{OPK}(h^+ q') u_0, \Xi_+^a v' \rangle; \end{aligned}$$

we have here used the rules for adjoints of Poisson and trace operators, and the fact that $\gamma_0^a u' = \gamma_1^{a-1} u + a \langle D' \rangle u_0$, cf. (3.9). (Operators in y' -form in the right-hand side give operators in x' -form when transposed to the left-hand side.) Thus

$$I_2 + I_3 - I'_2 - I'_3 = \langle s_0 \gamma_1^{a-1} u, \gamma_0^{a-1} v \rangle - \langle s_0 \gamma_0^{a-1} u, \gamma_1^{a-1} v \rangle.$$

Finally,

$$I_4 - I'_4 = ((\mathcal{B} - \mathcal{B}'^*) u_0, v_0)_{L_2(\mathbb{R}^{n-1})} = ((\mathcal{B} - \mathcal{B}'^*) \gamma_0^{a-1} u, \gamma_0^{a-1} v)_{L_2(\mathbb{R}^{n-1})},$$

where $B = \mathcal{B} - \mathcal{B}'^*$ satisfies, with $b(x', \xi') = \mathbf{b}(x', \xi') - \overline{\mathbf{b}'}(x', \xi')$,

$$(4.20) \quad B = \text{OP}'(b(x', \xi')), \quad b(x', \xi') = \lim_{z_n \rightarrow 0+} \check{q}(x', 0, \xi', z_n) - \lim_{z_n \rightarrow 0-} \check{q}(x', 0, \xi', z_n),$$

the jump at $z_n = 0$ in the bounded part of $\check{q}(x', 0, \xi', z_n)$. Altogether, we find:

$$\langle r^+ P u, v \rangle - \langle u, r^+ P^* v \rangle = (s_0 \gamma_1^{a-1} u, \gamma_0^{a-1} v) - (s_0 \gamma_0^{a-1} u, \gamma_1^{a-1} v) + (B \gamma_0^{a-1} u, \gamma_0^{a-1} v),$$

with $L_2(\mathbb{R}^{n-1})$ -dualities in the right-hand side, containing a generally nontrivial ψ do B on \mathbb{R}^{n-1} . The formulation in (4.2) follows by inserting (3.3).

The last assertion is seen as in Theorem 3.2. \square

Remark 4.2. Theorem 3.2 shows that $B = 0$ in the simple case of $P = (1 - \Delta)^a$. More generally, for operators with principal symbol $|\xi|^{2a}$, the principal (first-order) symbol of B is zero, so B is of order 0.

Note moreover that if $q(x', 0, \xi) - s_0(x', 0)$ is $O(\langle \xi_n \rangle^{-2})$, hence integrable in ξ_n , for all ξ' , then $\check{q}(x', 0, \xi', z_n) - s_0(x', 0)\delta(z_n)$ is a continuous function of z_n , so $b(x', \xi') = 0$ and hence $B = 0$. In general, q is the sum of a part depending only on the principal and subprincipal terms in p and a part that is $O(\langle \xi \rangle^{-2})$; the latter part does not contribute to B .

To extend the result of Theorem 4.1 to a domain Ω with curved boundary, we shall use a suitable cover of $\overline{\Omega}$ by coordinate charts, and a suitable partition of unity. Such choices were described in [G16], and we recall them in the following remark.

Remark 4.3. $\overline{\Omega}$ has a finite cover by bounded open sets U_0, \dots, U_I with C^∞ -diffeomorphisms $\kappa_i: U_i \rightarrow V_i$, V_i bounded open in \mathbb{R}^n , such that $U_i^+ = U_i \cap \Omega$ is mapped to $V_i^+ = V_i \cap \mathbb{R}_+^n$ and $U_i' = U_i \cap \partial\Omega$ is mapped to $V_i' = V_i \cap \partial\overline{\mathbb{R}_+^n}$; as usual we write $\partial\overline{\mathbb{R}_+^n} = \mathbb{R}^{n-1}$. For any such cover there exists an associated partition of unity, namely a family of functions $\varrho_i \in C_0^\infty(U_i)$ taking values in $[0, 1]$ such that $\sum_{i=0, \dots, I} \varrho_i$ is 1 on a neighborhood of $\overline{\Omega}$.

When P is a ψ do on \mathbb{R}^n , its application to functions supported in U_i carries over to functions on V_i as a ψ do $\underline{P}^{(i)}$ defined by

$$(4.21) \quad \underline{P}^{(i)} v = P(v \circ \kappa_i) \circ \kappa_i^{-1}, \quad v \in C_0^\infty(V_i).$$

We shall use a convenient system of coordinate charts as described in [G16], Remark 4.3: Here $\partial\Omega$ is covered with coordinate charts $\kappa_i': U_i' \rightarrow V_i' \subset \mathbb{R}^{n-1}$, $i = 1, \dots, I_0$, and the κ_i will be defined on certain subsets of a tubular neighborhood $\Sigma_r = \{x' + t\nu(x') \mid x' \in \partial\Omega, |t| < r\}$, where $\nu(x') = (\nu_1(x'), \dots, \nu_n(x'))$ is the interior normal to $\partial\Omega$ at $x' \in \partial\Omega$, and r is taken so small that the mapping $x' + t\nu(x') \mapsto (x', t)$ is a diffeomorphism from Σ_r to $\partial\Omega \times]-r, r[$. For each i , κ_i is defined as the mapping $\kappa_i: x' + t\nu(x') \mapsto (\kappa_i'(x'), t)$ ($x' \in U_i'$). κ_i goes from U_i to V_i , where

$$(4.22) \quad U_i = \{x' + t\nu(x') \mid x' \in U_i', |t| < r\}, \quad V_i = V_i' \times]-r, r[.$$

These charts are supplied with a chart consisting of the identity mapping on an open set U_0 containing $\Omega \setminus \overline{\Sigma}_{r,+}$, with $\overline{U}_0 \subset \Omega$, to get a full cover of $\overline{\Omega}$.

Note that the normal $\nu(x')$ at $x' \in \partial\Omega$ is carried over to the normal $(0, 1)$ at $(\kappa'_i(x'), 0)$ when $x' \in U'_i$. The halfline $L_{x'} = \{x' + t\nu(x') \mid t \geq 0\}$ is the geodesic into Ω orthogonal to $\partial\Omega$ at x' (with respect to the Euclidean metric on \mathbb{R}^n), and there is a positive $r' \leq r$ such that for $0 < t < r'$, the distance $d(x)$ between $x = x' + t\nu(x')$ and $\partial\Omega$ equals t . Then t plays the role of d in the definition of expansions and boundary values of $u \in \mathcal{E}_{a-1}(\bar{\Omega})$ in [G15] (5.3)ff. (cf. also (3.1) above)

$$(4.23) \quad u = \frac{1}{\Gamma(a)} t^{a-1} u_0 + \frac{1}{\Gamma(a+1)} t^a u_1 + \frac{1}{\Gamma(a+2)} t^{a+1} u_2 + \dots \text{ for } t > 0, \quad u = 0 \text{ for } t < 0,$$

where the u_j are constant in t for $t < r'$; this serves to define the boundary values $\gamma_j^{a-1} u = \gamma_0 u_j (= u_j|_{t=0})$, $j = 0, 1, 2, \dots$. The definition extends to define the two first boundary values $\gamma_0^{a-1} u$ and $\gamma_1^{a-1} u$ when $u \in H^{(a-1)(s)}(\bar{\Omega})$ with $s > a + \frac{1}{2}$ (for the first boundary value, $s > a - \frac{1}{2}$ suffices). By comparison of (4.23) with t^{a-1} times the Taylor expansion of u/t^{a-1} in t , we also have:

$$(4.24) \quad \gamma_0^{a-1} u = \Gamma(a) \gamma_0(u/t^{a-1}), \quad \gamma_1^{a-1} u = \Gamma(a+1) \gamma_1(u/t^{a-1}) = \Gamma(a+1) \gamma_0(\partial_t(u/t^{a-1})),$$

similarly as in (3.3).

In addition to the above construction of a cover by coordinate charts and an associated partition of unity, it is for some purposes practical to have a partition of unity *subordinate to a cover* as in [G16] Lemma 4.4. The cover is constructed from the cover we have just described, by an augmentation by extra coordinate charts $\kappa_i: U_i \rightarrow V_i$, $i = I_0 + 1, \dots, I_1$, such that there is a partition of unity ϱ_k , $k = 1, \dots, J_0$, where for any two functions ϱ_k, ϱ_l there is an $i = i(k, l)$ in $\{0, 1, \dots, I_1\}$ for which $\text{supp } \varrho_k \cup \text{supp } \varrho_l \subset U_{i(k, l)}$, see details in [G16]. The maps $\kappa_i: U_i \rightarrow V_i$ still have the property that the normal coordinate t at the boundary goes over into x_n (without distortion) for small t .

Theorem 4.4. *Let P is a classical ψ do on \mathbb{R}^n of order $2a > 0$ with even symbol, and let Ω be a smooth bounded subset. The following Green's formula holds for $u, v \in H^{(a-1)(s)}(\bar{\Omega})$ when $s > a + \frac{1}{2}$:*

$$(4.25) \quad \begin{aligned} & \langle r^+ P u, v \rangle_{\bar{H}^{-a+\frac{1}{2}+\varepsilon}(\Omega), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\bar{\Omega})} - \langle u, r^+ P^* v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\bar{\Omega}), \bar{H}^{-a+\frac{1}{2}+\varepsilon}(\Omega)} \\ & = (s_0 \gamma_1^{a-1} u, \gamma_0^{a-1} v) - (s_0 \gamma_0^{a-1} u, \gamma_1^{a-1} v) + (B \gamma_0^{a-1} u, \gamma_0^{a-1} v); \end{aligned}$$

the scalar products in the right-hand side are in $L_2(\partial\Omega)$, $s_0(x) = p_0(x, \nu(x))$ at boundary points x , and B is a first-order ψ do on $\partial\Omega$, depending only on the principal and subprincipal symbols of P .

Here the right-hand side equals, in terms of the normal coordinate t (with $\gamma_1 w = \gamma_0 \partial_t w$):

$$(4.26) \quad \Gamma(a) \Gamma(a+1) \int_{\partial\Omega} (s_0 \gamma_1(\frac{u}{t^{a-1}}) \gamma_0(\frac{\bar{v}}{t^{a-1}}) - s_0 \gamma_0(\frac{u}{t^{a-1}}) \gamma_1(\frac{\bar{v}}{t^{a-1}}) + a B \gamma_0(\frac{u}{t^{a-1}}) \gamma_0(\frac{\bar{v}}{t^{a-1}})) d\sigma.$$

When $s \geq 2a$, the left-hand side can be written as an integral over Ω .

Proof. For this proof we will use the cover U_i , $i = 0, \dots, I_1$, and the subordinate partition of unity ϱ_k , $k = 1, \dots, J_0$, described at the end of Remark 4.3. We can moreover choose real functions $\psi_k \in C_0^\infty(U_i)$ such that $\psi_k \varrho_k = \varrho_k$, i.e., ψ_k is 1 on $\text{supp } \varrho_k$. Then, with

$$(4.27) \quad u_k = \varrho_k u, \quad v_l = \varrho_l v, \quad P_{kl} = \psi_l P \psi_k, \quad P_{kl}^* = \psi_k P^* \psi_l,$$

we can write

$$(4.28) \quad \langle r^+ P u, v \rangle_\Omega - \langle u, r^+ P^* v \rangle_\Omega = \sum_{k, l \leq J_0} (\langle r^+ P_{kl} u_k, v_l \rangle_{U_i \cap \Omega} - \langle u_k, r^+ P_{kl}^* v \rangle_{U_i \cap \Omega}).$$

Recall the notation $U_i \cap \Omega = U_i^+$, $V_i \cap \mathbb{R}_+^n = V_i^+$. For each pair (k, l) we treat the term by use of the coordinate map for U_i , $i = i(k, l)$. Denote by \underline{P}_{kl} the operator on $V_i \subset \mathbb{R}^n$ that P_{kl} carries over to; its kernel is compactly supported in $V_i \times V_i$. In detail,

$$(4.29) \quad \underline{P}_{kl} = \underline{\psi}_l^{(i)} \underline{P}^{(i)} \underline{\psi}_k^{(i)},$$

cf. (4.21). The parity property of the symbol, hence the a -transmission property at any boundary, is preserved under the coordinate transformation.

In the i 'th term, the two sides are compactly supported in U_i . We shall need to carry this over to a (sesquilinear) distribution duality over V_i using the coordinate change κ_i , and therefore recall some general rules:

$$\begin{aligned} \langle f, g \rangle_{U_i} &= \langle \underline{f}, J \underline{g} \rangle_{V_i} \\ \langle T f, g \rangle_{U_i} &= \langle f, T^* g \rangle_{U_i} = \langle \underline{f}, J \underline{T^* g} \rangle_{V_i} \\ \langle T f, g \rangle_{U_i} &= \langle \underline{T f}, J \underline{g} \rangle_{V_i} = \langle \underline{f}, (\underline{T})^{(*)} (J \underline{g}) \rangle_{V_i} \\ &= \langle f, J^{-1} [(\underline{T})^{(*)} (J \underline{g})] \circ \kappa_i \rangle_{U_i}. \end{aligned}$$

Here the underlined objects are the elements carried over to V_i , and J is the Jacobian of the mapping κ_i^{-1} (the absolute value of its functional determinant); it is a smooth positive function. The star indicates the adjoint with respect to integration over U_i , and the star in parentheses indicates the adjoint with respect to integration over V_i . We see that the two concepts of adjoints are related by

$$J \underline{T^* g} = (\underline{T})^{(*)} (J \underline{g}).$$

We have for the (k, l) 'th term:

$$\langle r^+ P_{kl} u_k, v_l \rangle_{U_i^+} = \langle r^+ \underline{P}_{kl} \underline{u}_k, J \underline{v}_l \rangle_{V_i^+}, \quad \langle u_k, r^+ P_{kl}^* v_l \rangle_{U_i^+} = \langle \underline{u}_k, r^+ J (\underline{P}_{kl}^*) \underline{v}_l \rangle_{V_i^+},$$

so with $\underline{w}_l = J \underline{v}_l$,

$$(4.30) \quad \langle r^+ P_{kl} u_k, v_l \rangle_{U_i^+} - \langle u_k, r^+ P_{kl}^* v_l \rangle_{U_i^+} = \langle r^+ \underline{P}_{kl} \underline{u}_k, \underline{w}_l \rangle_{V_i^+} - \langle \underline{u}_k, r^+ (\underline{P}_{kl})^{(*)} \underline{w}_l \rangle_{V_i^+},$$

where $(\underline{P}_{kl})^{(*)}$ is the adjoint in the V_i^+ -setting,

$$(\underline{P}_{kl})^{(*)} = J (\underline{P}_{kl}^*) J^{-1}.$$

The underlined operators are defined on V_i , where $\kappa_i(U_i \cap \overline{\Omega}) \subset \overline{\mathbb{R}_+^n}$ and the relevant part of the boundary of $V_i \cap \overline{\mathbb{R}_+^n}$ is a subset of \mathbb{R}^{n-1} (all objects are supported away from the other parts of the boundary).

We can consider $\underline{P}^{(i)}$ as extended to an operator of the same type on all of \mathbb{R}^n with global estimates (and keep the name $\underline{P}^{(i)}$), so that (4.29) holds with the globally defined operator; this is followed up for the adjoint.

Now we are in a situation to apply Theorem 4.1, which gives

$$(4.31) \quad \begin{aligned} \langle r^+ \underline{P}_{kl} \underline{u}_k, \underline{w}_l \rangle_{V_i^+} - \langle \underline{u}_k, r^+ (\underline{P}_{kl})^{(*)} \underline{w}_l \rangle_{V_i^+} &= \langle r^+ \underline{P}_{kl} \underline{u}_k, \underline{w}_l \rangle_{\mathbb{R}_+^n} - \langle \underline{u}_k, r^+ (\underline{P}_{kl})^{(*)} \underline{w}_l \rangle_{\mathbb{R}_+^n} \\ &= (\underline{s}_{kl,0} \gamma_1^{a-1} \underline{u}_k, \gamma_0^{a-1} \underline{w}_l) - (\underline{s}_{kl,0} \gamma_0^{a-1} \underline{u}_k, \gamma_1^{a-1} \underline{w}_l) + (\underline{B}_{kl} \gamma_0^{a-1} \underline{u}_k, \gamma_0^{a-1} \underline{w}_l), \end{aligned}$$

the last line consists of $L_2(\mathbb{R}^{n-1})$ -scalar products. Here $\underline{s}_{kl,0} = \underline{\psi}_l \underline{p}_0(x', 0, 0, 1) \underline{\psi}_k$.

The dualities over V_i^+ carry over to dualities over U_i^+ by (4.30). For the scalar products over \mathbb{R}^{n-1} (supported in V_i'), we note that the coordinate transform preserves the definitions of γ_0^{a-1} and γ_1^{a-1} (since t corresponds exactly to x_n for $t < r'$),

$$[\gamma_0^{a-1} \underline{u}_k] \circ \kappa'_i = \gamma_0^{a-1} u_k, \quad [\gamma_1^{a-1} \underline{u}_k] \circ \kappa'_i = \gamma_1^{a-1} u_k.$$

Note here that $\gamma_0^{a-1} \underline{w}_l = J' \gamma_0^{a-1} \underline{v}_l$ and $\gamma_1^{a-1} \underline{w}_l = J' \gamma_1^{a-1} \underline{v}_l$ where J' is the restriction of J to the boundary (J is constant in t for small t , and J' equals the Jacobian for $(\kappa'_i)^{-1}$). Then (4.31) carries over to the formula on U_i^+ :

$$(4.32) \quad \begin{aligned} \langle r^+ P_{kl} u_k, v_l \rangle_{U_i^+} - \langle u_k, r^+ P_{kl}^* v_l \rangle_{U_i^+} \\ = (s_{kl,0} \gamma_1^{a-1} u_k, \gamma_0^{a-1} v_l) - (s_{kl,0} \gamma_0^{a-1} u_k, \gamma_1^{a-1} v_l) + (B_{kl} \gamma_0^{a-1} u_k, \gamma_0^{a-1} v_l), \end{aligned}$$

where $s_{kl,0} = \psi_l p_0(x, \nu(x)) \psi_k$ and the scalar products are over $L_2(\partial\Omega)$. Here B_{kl} depends only on the principal and subprincipal symbols of P_{kl} , cf. Remark 4.2.

Finally, summing over k and l and using that $\psi_k \varrho_k = \varrho_k$, we find (4.25) with

$$(4.33) \quad B \gamma_0^{a-1} u = \sum_{k,l} \varrho_l [\underline{B}_{kl} \underline{\varrho}_k \gamma_0^{a-1} u] \circ \kappa'_{i(k,l)}.$$

The last statements are seen as in Theorem 3.2. \square

Corollary 4.5. *Let P and Ω be as in Theorem 4.3.*

1° *If $u \in H^{(a-1)(s)}(\overline{\Omega})$, $v \in H^{a(s)}(\overline{\Omega})$, $s > a + \frac{1}{2}$, then*

$$(4.34) \quad \begin{aligned} \langle r^+ P u, v \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}(\Omega), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\Omega})} - \langle u, r^+ P^* v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\Omega}), \overline{H}^{-a+\frac{1}{2}+\varepsilon}(\Omega)} \\ = -(s_0 \gamma_0^{a-1} u, \gamma_0^a v)_{L_2(\partial\Omega)} \\ = -\Gamma(a) \Gamma(a+1) \int_{\partial\Omega} s_0 \gamma_0 \left(\frac{u}{t^{a-1}} \right) \gamma_0 \left(\frac{\bar{v}}{t^a} \right) d\sigma. \end{aligned}$$

2° *It follows that if $u, v \in H^{a(s)}(\overline{\Omega})$, $s > a + \frac{1}{2}$, then for each $j = 1, \dots, n$,*

$$(4.35) \quad \begin{aligned} \langle r^+ P u, \partial_j v \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}(\Omega), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\Omega})} + \langle \partial_j u, r^+ P^* v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\Omega}), \overline{H}^{-a+\frac{1}{2}+\varepsilon}(\Omega)} \\ = (\nu_j s_0 \gamma_0^a u, \gamma_0^a v)_{L_2(\partial\Omega)} + (r^+ [P, \partial_j] u, v)_{L_2(\Omega)} \\ = \Gamma(a+1)^2 \int_{\partial\Omega} \nu_j s_0 \gamma_0 \left(\frac{u}{t^a} \right) \gamma_0 \left(\frac{\bar{v}}{t^a} \right) d\sigma + (r^+ [P, \partial_j] u, v)_{L_2(\Omega)}. \end{aligned}$$

The dualities in the left-hand sides can be written as integrals when $s \geq 2a$.

Proof. 1° follows simply by application of Theorem 4.4 with $\gamma_0^{a-1}v = 0$; then $\gamma_1^{a-1}v = \gamma_0^av$, as noted earlier.

2° is deduced from this as follows: First we observe that

$$\langle r^+Pu, \partial_j v \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}, \dot{H}^{a-\frac{1}{2}-\varepsilon}} = -\langle \partial_j r^+Pu, v \rangle_{\overline{H}^{-a-\frac{1}{2}+\varepsilon}, \dot{H}^{a+\frac{1}{2}-\varepsilon}}$$

by integration by parts, using that $\gamma_0 v = 0$ ($v \in H^{a(\frac{1}{2}+\varepsilon)}(\overline{\Omega}) \subset \dot{H}^{a+\frac{1}{2}-\varepsilon}(\overline{\Omega})$). Next, we introduce the commutator $[P, \partial_j] = P\partial_j - \partial_j P$. Now $\partial_j u \in H^{(a-1)(s)}$, and (4.34) applies: (4.36)

$$\begin{aligned} & \langle r^+Pu, \partial_j v \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}, \dot{H}^{a-\frac{1}{2}-\varepsilon}} + \langle \partial_j u, r^+P^*v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}, \overline{H}^{-a+\frac{1}{2}+\varepsilon}} \\ &= -\langle r^+P\partial_j u, v \rangle_{\overline{H}^{-a-\frac{1}{2}+\varepsilon}, \dot{H}^{a+\frac{1}{2}-\varepsilon}} + \langle \partial_j u, r^+P^*v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}, \overline{H}^{-a+\frac{1}{2}+\varepsilon}} + (r^+[P, \partial_j]u, v) \\ &= (s_0\gamma_0^{a-1}(\partial_j u), \gamma_0^av) + (r^+[P, \partial_j]u, v). \end{aligned}$$

Write u near $\partial\Omega$ as $u = t^aw(y' + \nu(y')t)$, where $y' \in \partial\Omega$ and w is constant in t for $0 \leq t < r'$, and use that $\partial_j = \nu_j(y')\partial_t + T$, where T is tangential (acts along $\partial\Omega$) there, to see that

$$\partial_j u = \nu_j(y')\partial_t(t^aw) + T(t^aw) = \nu_j a t^{a-1}w + t^a T w,$$

hence

$$\begin{aligned} \gamma_0^{a-1}(\partial_j u) &= \Gamma(a)\gamma_0(\partial_j u/t^{a-1}) = \Gamma(a)\gamma_0(\nu_j a w + t T w) \\ &= \Gamma(a+1)\nu_j \gamma_0 w = \Gamma(a+1)\nu_j \gamma_0(u/t^a) = \nu_j \gamma_0^a u. \end{aligned}$$

Insertion in (4.36) leads to (4.35). \square

A version of 1° in the corollary was shown by Abatangelo [A15], see (9), in the case $P = (-\Delta)^a$, $0 < a < 1$. Since d^{a-1} blows up for $d \rightarrow 0$ in this case, the functions u entering in the formula are called “large solutions” in [A15].

2° was shown by Ros-Oton and Serra for $(-\Delta)^a$, $0 < a < 1$, in [RS14b] (under different smoothness hypotheses), extended to higher-order fractional Laplacians in [RS15], and generalized to larger classes of translation-invariant operators in a joint work with Valdinoci [RSV16]. We extended it to (x -dependent) elliptic ψ do's in [G16]. It implies Pohozaev formulas that are used to show uniqueness results for nonlinear problems.

Note that no assumption on ellipticity of P is made in the theorem and corollary. Actually, it is not surprising that ellipticity is not needed, since the formulas are linear in P : For a given P one can add $c(1 - \Delta)^a$ with a sufficiently large constant c to make the sum P_c strongly elliptic; a result for P_c will then lead to a result for P by subtraction of the formula for $c(1 - \Delta)^a$. However, ellipticity was an important ingredient in earlier proofs of the corollary, that we can now do away with.

The corollary only involves first boundary values, and the boundary contribution is purely local. Theorem 4.4, however, lets the Neumann value enter in a nontrivial way along with a nontrivial Dirichlet value; this seems to be entirely new even for $(-\Delta)^a$.

5. A parametrix of the Dirichlet problem.

Green's formula shows that the Dirichlet and Neumann trace operators γ_0^{a-1} and γ_1^{a-1} play a fundamental role in the discussion of boundary value problems for P . Much is known about the homogeneous Dirichlet problem (1.5), whereas problems for P with nonzero Dirichlet trace have been less studied. We showed the Fredholm solvability in [G15, G14] for large scales of spaces over a smooth bounded set Ω . The structure of the solution operator will now be further clarified, in a study of the operator K_D solving (1.9).

In addition to the assumptions listed in the beginning of Section 4, we assume that P is elliptic, the principal symbol avoiding a ray. This holds in particular if P is strongly elliptic, i.e.

$$(5.1) \quad \operatorname{Re} p_0(x, \xi) \geq c(x)|\xi|^{2a} \text{ for } |\xi| \geq 1, x \in \mathbb{R}^n, \text{ with } c(x) > 0;$$

it is said to be uniformly strongly elliptic when $c(x)$ can be taken constant.

The following mapping was defined in [G15] for $s > a - \frac{1}{2}$,

$$(5.2) \quad \begin{pmatrix} r^+ P \\ \gamma_0^{a-1} \end{pmatrix} : H^{(a-1)(s)}(\overline{\Omega}) \rightarrow \overline{H}^{s-2a}(\Omega) \times H^{s-a+\frac{1}{2}}(\partial\Omega);$$

by Th. 6.1 there it is Fredholm when Ω is bounded. Extensions of the mapping property to H_p^s -spaces and other scales of spaces are given in [G14, G15]. To exhibit a bijective case, we show:

Lemma 5.1. *Let P be a classical ψ do on \mathbb{R}^n of order $2a > 0$, strongly elliptic with even symbol on \mathbb{R}^n , and let Ω be a smooth bounded subset. If P has a positive lower bound on $C_0^\infty(\Omega)$:*

$$(5.3) \quad \operatorname{Re}(Pu, u)_{L_2(\Omega)} \geq c_0 \|u\|_{L_2(\Omega)}^2 \text{ for } u \in C_0^\infty(\Omega),$$

with $c_0 > 0$, then the mapping (5.2) is bijective. The solution operator, denoted $(R_D \quad K_D)$, maps as follows:

$$(5.4) \quad \begin{pmatrix} r^+ P \\ \gamma_0^{a-1} \end{pmatrix}^{-1} = (R_D \quad K_D) : \overline{H}^{s-2a}(\Omega) \times H^{s-a+\frac{1}{2}}(\partial\Omega) \rightarrow H^{(a-1)(s)}(\overline{\Omega}), \quad s > a - \frac{1}{2}.$$

Proof. As mentioned in Section 1, the Dirichlet realization P_D defined by the variational construction (the Lax-Milgram lemma) acts like $r^+ P$ with domain

$$(5.5) \quad D(P_D) = \{u \in \dot{H}^a(\overline{\Omega}) \mid r^+ Pu \in L_2(\Omega)\}.$$

It was found in [G15] that $D(P_D) = H^{a(2a)}(\overline{\Omega})$. The adjoint is the analogous operator for P^* , which also satisfies (5.3). This inequality assures that both P_D and P_D^* are injective, hence P_D is bijective from $H^{a(2a)}(\overline{\Omega})$ to $L_2(\Omega)$. Now $H^{a(2a)}(\overline{\Omega}) = \{u \in H^{(a-1)(2a)}(\overline{\Omega}) \mid \gamma_0^{a-1}u = 0\}$, and

$$(5.6) \quad \gamma_0^{a-1} : H^{(a-1)(2a)}(\overline{\Omega}) / H^{a(2a)}(\overline{\Omega}) \xrightarrow{\sim} H^{a+\frac{1}{2}}(\partial\Omega)$$

(cf. [G15], Th. 5.1), so the bijectiveness in (5.2) for $s = 2a$ follows by supplying the mapping P_D with (5.6). The bijectiveness holds also for other s in view of the invariance of kernels and cokernels ([G14], Th. 5.5). \square

An example of an operator satisfying the hypotheses of Lemma 5.1 is $(1 - \Delta)^a$, since

$$((1 - \Delta)^a u, u) = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{2a} |\hat{u}(\xi)|^2 d\xi \geq \|u\|_{L_2(\mathbb{R}^n)}^2 \text{ for all } u \in \mathcal{S}(\mathbb{R}^n).$$

In view of (3.12)ff. and (3.14), the operator K_D for $P = (1 - \Delta)^a$ on \mathbb{R}_+^n equals K_0^{a-1} .

In [G15] we obtained in Th. 4.4 that

$$(5.7) \quad R_D = \Lambda_+^{(-a)} e^+ \widetilde{Q_+} r^+ \Lambda_-^{(-a)} e^+ : \overline{H}^{s-2a}(\Omega) \rightarrow H^{(a)(s)}(\overline{\Omega})$$

is a parametrix of the Dirichlet problem with zero boundary condition; here $Q = \Lambda_-^{(-a)} P \Lambda_+^{(-a)}$, $Q_+ = r^+ Q e^+ : \overline{H}^t(\Omega) \rightarrow \overline{H}^t(\Omega)$ (for all $t > -\frac{1}{2}$), and $\widetilde{Q_+}$ is a parametrix of Q_+ . Moreover, we showed in Th. 6.5 there how, in the case $\Omega = \mathbb{R}_+^n$, R_D could be supplied with a Poisson-like operator

$$K_D : H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1}) \rightarrow H^{(a-1)(s)}(\overline{\mathbb{R}_+^n})$$

constructed from R_D , to give a full parametrix. The operator K_D was shown to be of the form

$$K_D = \Xi_+^{1-a} e^+ K' = \Lambda_+^{1-a} e^+ K'',$$

with K' and K'' being Poisson operators of order 0 belonging to the Boutet de Monvel calculus.

In [G16] Th. 2.7, assuming that P is elliptic avoiding a ray, with even symbol, we worked out an approximate factorization of P in "minus" and "plus" operators (preserving support in $\overline{\mathbb{R}_-^n}$ resp. $\overline{\mathbb{R}_+^n}$) in the case $\Omega = \mathbb{R}_+^n$,

$$(5.8) \quad P \sim P^- P^+,$$

and we indicated in Rem. 2.9 there a formula for K_D based on the factorization, namely essentially

$$(5.9) \quad K_D \varphi \sim r^+ \widetilde{P}^+ (\varphi(x') \otimes \delta(x_n)),$$

where \widetilde{P}^+ is a parametrix of P^+ . We shall now go into details with this construction, and thereby also obtain a more informative formula for the full parametrix $(R_D \quad K_D)$.

As accounted for in the Appendix, we have the following product decompositions of P and its parametrix \widetilde{P} :

$$(5.10) \quad \begin{aligned} P &= \Xi_-^a Q \Xi_+^a = \Lambda_-^a Q_1 \Lambda_+^a \sim P^- P^+, \text{ where} \\ Q &= \Xi_-^a P \Xi_+^{-a} \sim Q^- Q^+, \quad Q_1 = \Lambda_-^a P \Lambda_+^{-a} \sim Q_1^- Q_1^+, \text{ and} \\ P^- &= \Xi_-^a Q^- = \Lambda_-^a Q_1^-, \quad P^+ = Q^+ \Xi_+^a = Q_1^+ \Lambda_+^a; \text{ moreover,} \\ \widetilde{P} &\sim \widetilde{P}^+ \widetilde{P}^-, \quad \widetilde{P}^+ = \Xi_-^a \widetilde{Q}^+ = \Lambda_-^a \widetilde{Q}_1^+, \quad \widetilde{P}^- = \widetilde{Q}^- \Xi_-^a = \widetilde{Q}_1^- \Lambda_-^a. \end{aligned}$$

In view of the support-preserving properties, when $\text{supp } u \subset \overline{\mathbb{R}}_+^n$, $P^+u \in \dot{H}^{-\frac{1}{2}+\varepsilon}(\overline{\mathbb{R}}_+^n)$,

$$(5.11) \quad r^+Pu \sim r^+P^-P^+u = r^+P^-e^+r^+P^+u,$$

with similar rules for Q and Q_1 .

Let us analyse how K_D should look, by an argumentation that reduces the problem as far as possible to the Boutet de Monvel calculus, in the same spirit as the proof of Th. 4.4 in [G15]. (One advantage of that calculus is that there are good rules for smoothing operators, which can be difficult to obtain in a mixture of fractional-order ψ do's and the generalized ψ do's that in some situations only give good tangential results.)

We want to solve:

$$(5.12) \quad r^+Pu = 0, \quad \gamma_0^{a-1}u = \varphi,$$

for a given $\varphi \in H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1})$ with $s > a - \frac{1}{2}$, searching for u in $H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n)$. Let $\tilde{u} \in \overline{H}^{s-a+1}(\mathbb{R}_+^n)$ be the function for which

$$(5.13) \quad \tilde{u} = r^+\Xi_+^{1-a}u; \text{ then } u = \Xi_+^{a-1}e^+\tilde{u} \text{ and } e^+\tilde{u} = \Xi_+^{1-a}u,$$

by definition (cf. [G15], Sect. 1). For example when $s = a$, $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$ and $\tilde{u} \in \overline{H}^1(\mathbb{R}_+^n)$.

By use of (5.10), (5.11), the equations in (5.12) can be written

$$r^+\Xi_-^ae^+r^+Q\Xi_+^1\Xi_+^{a-1}u = 0, \quad \gamma_0\Xi_+^{a-1}u = \varphi.$$

Using (5.13), this may, since $r^+\Xi_-^ae^+$ is bijective, equivalently be written

$$(5.14) \quad r^+Q\Xi_+^1e^+\tilde{u} = 0, \quad \gamma_0\tilde{u} = \varphi,$$

where the boundary value is as always taken from \mathbb{R}_+^n .

Now (5.14) is concerned with integer-order operators, where we have the rules of the Boutet de Monvel calculus, extended to allow the generalized ψ do's Q^+ and Ξ_+^1 . Here we note that $r^+Q\Xi_+^1e^+\tilde{u} \sim r^+Q^-e^+r^+Q^+\Xi_+^1e^+\tilde{u}$, where $r^+Q^-e^+$ has the parametrix $r^+\tilde{Q}^-e^+$. This allows transforming (5.14) to

$$(5.15) \quad r^+Q^+\Xi_+^1e^+\tilde{u} \sim 0, \quad \gamma_0\tilde{u} = \varphi,$$

in a parametrix sense. Here, since $Q^+\Xi_+^1$ has the parametrix $\Xi_+^{-1}\tilde{Q}^+$, the problem (5.15) can be expected to have the solution operator

$$(5.16) \quad K_{Q^+}\varphi \equiv r^+\Xi_+^{-1}\tilde{Q}^+(\varphi(x') \otimes \delta(x_n)),$$

in a parametrix sense. This is the Poisson operator with symbol $k_{Q^+} \in S^{-1}(\mathcal{H}^+)$,

$$(5.17) \quad k_{Q^+}(x', \xi) = \chi_+^{-1}(\xi) \# \tilde{q}^{+'}(x', 0, \xi),$$

by the rule (A.13), the prime indicating that we have used the (x', y_n) -form of the symbol of \tilde{Q}^+ , cf. (A.11). It is known from the calculus that K_{Q^+} maps

$$(5.18) \quad K_{Q^+}: H^{t-\frac{1}{2}}(\mathbb{R}^{n-1}) \rightarrow \overline{H}^t(\mathbb{R}_+^n) \text{ for } t \in \mathbb{R}.$$

In case $Q = I$, K_{Q^+} is the standard Poisson operator $K_0 = \text{OPK}(\chi_+^{-1})$ which satisfies $\gamma_0K_0 = I$.

With this solution operator to (5.15), we go on to define the solution operator for (5.12) by

$$(5.19) \quad K_D = \Xi_+^{1-a}e^+K_{Q^+}: H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1}) \rightarrow \Xi_+^{1-a}e^+\overline{H}^{s-a+1}(\mathbb{R}_+^n) = H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n).$$

It is accounted for how this K_D has the desired parametrix property in the proof of the following theorem.

Theorem 5.2. *Let P be a classical globally estimated ψ do on \mathbb{R}^n of order $2a > 0$, elliptic avoiding a ray, with even symbol.*

1° *Define K_D on $H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1})$, $s > a - \frac{1}{2}$, by (5.19) with (5.16). It solves the Dirichlet problem*

$$(5.20) \quad r^+ P u = 0 \text{ on } \mathbb{R}_+^n, \quad \gamma_0^{a-1} u = \varphi \text{ at } x_n = 0,$$

in a parametrix sense, namely

$$(5.21) \quad \gamma_0^{a-1} K_D = I, \quad r^+ P K_D = \mathcal{S}: H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1}) \rightarrow C^\infty(\overline{\mathbb{R}}_+^n).$$

2° *Define instead K_D by*

$$(5.22) \quad K_D = \Lambda_+^{1-a} e^+ K_{Q_+}, \text{ where } K_{Q_+} \varphi = r^+ \Lambda_+^{-1} \tilde{Q}_1^+(\varphi(x') \otimes \delta(x_n)).$$

Then we again have that K_D solves (5.20) in a parametrix sense, namely (5.21) holds (with a possibly different \mathcal{S}).

Proof. 1°. We begin by checking that K_{Q_+} defined in (5.16) solves the problem (5.15). First,

$$r^+ Q^+ \Xi_+^{-1} e^+ K_{Q_+} \varphi = r^+ Q^+ \Xi_+^{-1} e^+ r^+ \Xi_+^{-1} \tilde{Q}^+(\varphi \otimes \delta) = r^+ Q^+ \tilde{Q}^+(\varphi \otimes \delta) = \mathcal{S}_1 \varphi,$$

where \mathcal{S}_1 is a Poisson operator of order $-\infty$, hence maps $H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1})$ into $C^\infty(\overline{\mathbb{R}}_+^n)$ (or more precisely, into the subset $\bigcap_t \overline{H}^t(\mathbb{R}_+^n)$). This uses that $(I - e^+ r^+)v = 0$ when $v = \Xi_+^{-1} \tilde{Q}^+(\varphi \otimes \delta)$, since it is an L_2 -function supported in $\overline{\mathbb{R}}_+^n$; then since $Q^+ \tilde{Q}^+ = I + \mathcal{R}$, where \mathcal{R} has symbol $r(x', \xi) \in S^{-\infty}(\mathcal{H}^+)$, \mathcal{S}_1 equals $\text{OPK}(r)$, a smoothing Poisson operator.

Next, since \tilde{Q}^+ has a symbol of the form $1 + f^+$, where $f^+ = h^+ \tilde{q}^+ \in S^0(\mathcal{H}^+)$,

$$\gamma_0 K_{Q_+} \varphi = \gamma_0 \Xi_+^{-1} (I + F^+) (\varphi \otimes \delta) = \gamma_0 K_0 \varphi + \gamma_0 \Xi_+^{-1} F^+ (\varphi \otimes \delta) = \varphi,$$

where the term with $\Xi_+^{-1} F^+$ gives zero since the symbol is in \mathcal{H}^+ w.r.t. ξ_n and is $O(\xi_n^{-2})$, so that its integral in ξ_n vanishes. Altogether,

$$(5.23) \quad r^+ Q^+ \Xi_+^{-1} e^+ K_{Q_+} = \mathcal{S}_1, \quad \gamma_0 K_{Q_+} = I.$$

Now define K_D by (5.19). Then

$$(5.24) \quad r^+ P K_D \varphi = r^+ \Xi_-^a e^+ r^+ Q \Xi_+^a K_D \varphi = r^+ \Xi_-^a e^+ r^+ (Q^- Q^+ + \mathcal{R}_1) \Xi_+^{-1} e^+ K_{Q_+} \varphi,$$

where \mathcal{R}_1 has symbol in $S^{-\infty}(\mathcal{H}_{-1})$. Here

$$r^+ Q^- Q^+ \Xi_+^{-1} e^+ K_{Q_+} \varphi = r^+ Q^- e^+ r^+ Q^+ \Xi_+^{-1} e^+ K_{Q_+} \varphi = r^+ Q^- e^+ \mathcal{S}_1 \varphi = \mathcal{S}_2 \varphi,$$

in view of (5.23), where \mathcal{S}_2 maps into $\bigcap_t \overline{H}^t(\mathbb{R}_+^n)$, since $r^+ Q^- e^+$ preserves this space (by [G16] Th. 2.4 combined with support-preserving properties). Composition with $r^+ \Xi_-^a e^+$

to the left gives another smoothing operator \mathcal{S}_3 in view of the isomorphism properties of $r^+\Xi_-^a e^+$. The contribution from \mathcal{R}_1 in (5.24) equals

$$r^+\Xi_-^a e^+ r^+ \mathcal{R}_1 \Xi_+^1 e^+ K_{Q^+} \varphi = \mathcal{S}_4 \varphi,$$

where \mathcal{S}_4 is smoothing, since $r^+ \mathcal{R}_1 \Xi_+^1 e^+ K_{Q^+}$ is a Poisson operator with symbol in $S^{-\infty}(\mathcal{H}^+)$. Concerning the boundary condition, we note that

$$\gamma_0^{a-1} K_D \varphi = \gamma_0 \Xi_+^{a-1} K_D \varphi = \gamma_0 \Xi_+^{a-1} \Xi_+^{1-a} e^+ K_{Q^+} \varphi = \gamma_0 K_{Q^+} \varphi = \varphi,$$

in view of (5.23). Taking $\mathcal{S} = \mathcal{S}_3 + \mathcal{S}_4$, we have obtained 1°.

2°. We know from [G15] that $H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n)$ can equivalently be defined as $\Lambda_+^{1-a} e^+ \overline{H}^{s-a+1}(\mathbb{R}_+^n)$. Moreover, by (A.15),

$$(5.25) \quad \gamma_0^{a-1} u = \gamma_0(\Xi_+^{a-1} u) = \gamma_0(\Lambda_+^{a-1} u).$$

The Poisson operator $K'_0 = \text{OPK}(\lambda_+^{-1})$ satisfies $\gamma_0 K'_0 = I$, and $\Lambda_+^{1-a} e^+ K'_0$ is a right-inverse of γ_0^{a-1} .

The proof under 1° goes over verbatim to a proof of 2° when the Ξ_{\pm}^t -family is replaced by Λ_{\pm}^t , Q is replaced by Q_1 , and \tilde{u} is replaced by $\tilde{u}' = r^+ \Lambda_+^{a-1} u$. \square

Corollary 5.3. *Together with R_D recalled above, K_D enters in a full parametrix*

$$(5.26) \quad (R_D \quad K_D) : \overline{H}^{s-2a}(\mathbb{R}_+^n) \times H^{s-a+\frac{1}{2}}(\mathbb{R}^{n-1}) \rightarrow H^{(a-1)(s)}(\overline{\mathbb{R}}_+^n), \quad s > a - \frac{1}{2},$$

satisfying

$$(5.27) \quad \begin{pmatrix} r^+ P \\ \gamma_0^{a-1} \end{pmatrix} (R_D \quad K_D) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} \mathcal{S}_1 & \mathcal{S} \\ 0 & 0 \end{pmatrix},$$

where \mathcal{S}_1 maps $\overline{H}^{s-2a}(\mathbb{R}_+^n)$ into $C^\infty(\overline{\mathbb{R}}_+^n)$ and \mathcal{S} is as in (5.21).

Proof. From the way in which R_D was defined in [G15], we have that $\gamma_0^{a-1} R_D = 0$ and $r^+ P R_D = I + \mathcal{S}_1$ where \mathcal{S}_1 is a smoothing operator; (5.27) follows by combining this with the above theorem. \square

By use of local coordinates (cf. Remark 4.3), the above construction of K_D in the case \mathbb{R}_+^n can be applied to construct the Poisson-like operator in the general case of a bounded smooth open set Ω .

Theorem 5.4. *Let P be a classical ψ do of order $2a > 0$, elliptic avoiding a ray, with even symbol, and let Ω be a smooth bounded subset of \mathbb{R}^n . There is a Poisson-like operator $K_D : H^{s-a+\frac{1}{2}}(\partial\Omega) \rightarrow H^{(a-1)(s)}(\overline{\Omega})$, $s > a - \frac{1}{2}$ (see details in (5.31)), such that together with R_D constructed in [G15] Th. 4.4,*

$$(5.28) \quad (R_D \quad K_D) : \overline{H}^{s-2a}(\Omega) \times H^{s-a+\frac{1}{2}}(\partial\Omega) \rightarrow H^{(a-1)(s)}(\overline{\Omega}), \quad s > a - \frac{1}{2},$$

is a parametrix of the nonhomogeneous Dirichlet problem, satisfying

$$(5.29) \quad \begin{pmatrix} r^+ P \\ \gamma_0^{a-1} \end{pmatrix} \begin{pmatrix} R_D & K_D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} \mathcal{S}_1 & \mathcal{S} \\ 0 & 0 \end{pmatrix},$$

where $\mathcal{S}_1: \overline{H}^{s-2a}(\Omega) \rightarrow C^\infty(\overline{\Omega})$ and $\mathcal{S}: H^{s-a+\frac{1}{2}}(\partial\Omega) \rightarrow C^\infty(\overline{\Omega})$.

Proof. We shall here use the cover U_i , $i = 0, \dots, I_0$, described in Remark 4.3, with an associated partition of unity $\varrho_0, \dots, \varrho_{I_0}$ such that each ϱ_i is in $C_0^\infty(U_i)$ taking values in $[0, 1]$, and $\sum_{0 \leq i \leq I_0} \varrho_i(x) = 1$ on a neighborhood of $\overline{\Omega}$. One can moreover find functions $\zeta_i^0, \zeta_i^1, \zeta_i^2, \dots \in C_0^\infty(U_i, [0, 1])$ such that $\zeta_i^0 \varrho_i = \varrho_i$ and $\zeta_i^{k+1} \zeta_i^k = \zeta_i^k$ for all k (in each case, the former function is 1 on the support of the latter).

P considered on functions supported in U_i gives rise to $\underline{P}^{(i)}$ defined on functions supported in V_i by (4.21). The symbol of the operator $\underline{P}^{(i)}$ can be assumed to be extended to a ψ do symbol on \mathbb{R}^n , even and of order $2a$, and elliptic avoiding a ray there (we use the same notation for the extension). There are factorizations, as described in (5.10), with notation

$$(5.30) \quad \underline{P}^{(i)} = \Lambda_-^a \underline{Q}_1^{(i)} \Lambda_+^a, \quad \underline{Q}_1^{(i)} \sim \underline{Q}_1^{(i)-} \underline{Q}_1^{(i)+}, \text{ etc.}$$

For φ given on $\partial\Omega$, $\varphi = \sum_{0 \leq i \leq I_0} \varrho_i \varphi$, where $\varrho_i \varphi$ carries over to $\underline{\varrho}_i \varphi$ on V_i' . Define

$$(5.31) \quad \begin{aligned} K_{\underline{Q}_1^+}^{(i)} \psi &= r^+ \Lambda_+^{-1} \widetilde{\underline{Q}_1}^{(i)+} (\underline{\psi}(x') \otimes \delta(x_n)) \text{ when } \text{supp } \underline{\psi} \subset V_i', \\ \underline{K}_D^{(i)} \underline{\psi} &= \Lambda_+^{1-a} e^+ K_{\underline{Q}_1^+}^{(i)} \underline{\psi}, \\ K_D \varphi &= \sum_i (\zeta_i^1 \underline{K}_D^{(i)} \underline{\varrho}_i \varphi) \circ \kappa_i. \end{aligned}$$

(The contribution from U_0 is 0.) We shall verify that this definition of K_D leads to the desired properties.

For the composition with $r^+ P$ we have:

$$\begin{aligned} r^+ P K_D \varphi &= \sum_i r^+ P (\zeta_i^1 \underline{K}_D^{(i)} \underline{\varrho}_i \varphi) \circ \kappa_i \\ &= \sum_i r^+ \zeta_i^2 P (\zeta_i^1 \underline{K}_D^{(i)} \underline{\varrho}_i \varphi) \circ \kappa_i + \mathcal{S}_2 \varphi, \end{aligned}$$

with a smoothing operator \mathcal{S}_2 , since $1 - \zeta_i^2$ and ζ_i^1 have disjoint supports so that $(1 - \zeta_i^2) P \zeta_i^1$ is a ψ do of order $-\infty$.

Consider here the i 'th term carried over to V_i , omitting (i) from the notation for simplicity:

$$\begin{aligned} r^+ \zeta_i^2 \underline{P} \zeta_i^1 \underline{K}_D \underline{\varrho}_i \varphi &= r^+ \zeta_i^2 \Lambda_-^a e^+ r^+ \underline{Q}_1 \Lambda_+^a \zeta_i^1 \Lambda_+^{1-a} e^+ K_{\underline{Q}_1^+} \underline{\varrho}_i \varphi \\ &= r^+ \zeta_i^2 \Lambda_-^a e^+ r^+ \underline{Q}_1 \Lambda_+^a \Lambda_+^{1-a} e^+ K_{\underline{Q}_1^+} \underline{\varrho}_i \varphi + r^+ \zeta_i^2 \Lambda_-^a e^+ r^+ \underline{Q}_1 \Lambda_+^a (1 - \zeta_i^1) \Lambda_+^{1-a} e^+ K_{\underline{Q}_1^+} \underline{\varrho}_i \varphi. \end{aligned}$$

The first term equals

$$(5.32) \quad r^+ \underline{\zeta}_i^2 \Lambda_-^a e^+ r^+ \underline{Q}_1 \Lambda_+^1 e^+ K_{\underline{Q}_1^+} \underline{\varrho}_i \varphi,$$

and will be treated further below. The second term equals

$$\begin{aligned} & r^+ \underline{\zeta}_i^2 \Lambda_-^a e^+ r^+ \underline{Q}_1 \Lambda_+^a (1 - \underline{\zeta}_i^1) \Lambda_+^{1-a} e^+ r^+ \Lambda_+^{-1} \tilde{Q}_1^+ \underline{\zeta}_i^0 (\underline{\varrho}_i \varphi \otimes \delta) \\ &= r^+ \underline{\zeta}_i^2 \Lambda_-^a e^+ r^+ \underline{Q}_1 \Lambda_+^a (1 - \underline{\zeta}_i^1) \Lambda_+^{1-a} \Lambda_+^{-1} \tilde{Q}_1^+ \underline{\zeta}_i^0 (\underline{\varrho}_i \varphi \otimes \delta) \\ &= r^+ \underline{\zeta}_i^2 \Lambda_-^a e^+ r^+ \underline{Q}_1 \Lambda_+^a (1 - \underline{\zeta}_i^1) \Lambda_+^{-a} \tilde{Q}_1^+ \underline{\zeta}_i^0 (\underline{\varrho}_i \varphi \otimes \delta), \end{aligned}$$

where $\Lambda_+^a (1 - \underline{\zeta}_i^1) \Lambda_+^{-a}$ is a ψ do of order 0 having the 0-transmission property; we used in the proof that $(I - e^+ r^+) \Lambda_+^{-1} \tilde{Q}_1^+ \underline{\zeta}_i^0 (\underline{\varrho}_i \varphi \otimes \delta) = 0$. In this term,

$$(5.33) \quad \underline{\varrho}_i \varphi \mapsto r^+ \underline{Q}_1 \Lambda_+^a (1 - \underline{\zeta}_i^1) \Lambda_+^{-a} \tilde{Q}_1^+ \underline{\zeta}_i^0 (\underline{\varrho}_i \varphi \otimes \delta)$$

is a Poisson operator of order 0, derived from a generalized ψ do whose symbol contains the factors $1 - \underline{\zeta}_i^1$ and $\underline{\zeta}_i^0$ with disjoint supports; hence it must be a Poisson operator of order $-\infty$. Then it maps into $\bigcap_t \overline{H}^t(\mathbb{R}_+^n)$, and the composition with $r^+ \underline{\zeta}_i^2 \Lambda_-^a e^+$ maps into $r^+ C_0^\infty(V_i)$. In conclusion, this term reduces to a smoothing contribution $\mathcal{S}_{i,3}\varphi$.

We now continue with the term in (5.32). Here

$$\begin{aligned} r^+ \underline{Q}_1 \Lambda_+^1 e^+ K_{\underline{Q}_1^+} \underline{\varrho}_i \varphi &= r^+ \underline{Q}_1 \Lambda_+^1 e^+ r^+ \Lambda_+^{-1} \tilde{Q}_1^+ (\underline{\varrho}_i \varphi \otimes \delta) \\ &= r^+ (\underline{Q}_1^- \underline{Q}_1^+ + \mathcal{R}_1) \Lambda_+^1 \Lambda_+^{-1} \tilde{Q}_1^+ (\underline{\varrho}_i \varphi \otimes \delta) \\ &= r^+ (\underline{Q}_1^- \underline{Q}_1^+ + \mathcal{R}_1) \tilde{Q}_1^+ (\underline{\varrho}_i \varphi \otimes \delta), \end{aligned}$$

where \mathcal{R}_1 has symbol in $S^{-\infty}(\mathcal{H}_{-1})$. The contribution from \mathcal{R}_1 is seen as above to be smoothing, and

$$r^+ \underline{Q}_1^- \underline{Q}_1^+ \tilde{Q}_1^+ (\underline{\varrho}_i \varphi \otimes \delta) = r^+ \underline{Q}_1^- (I + \mathcal{R}_2) (\underline{\varrho}_i \varphi \otimes \delta),$$

where the contribution from \mathcal{R}_2 is likewise seen to be smoothing. Finally,

$$r^+ \underline{Q}_1^- (\underline{\varrho}_i \varphi \otimes \delta) = 0,$$

since \underline{Q}_1^- preserves support in $\overline{\mathbb{R}}_+^n$. Collecting the contributions, carried back to the coordinate patches U_i , we find that

$$(5.34) \quad r^+ P K_D \varphi = \mathcal{S} \varphi,$$

where \mathcal{S} maps into $C^\infty(\overline{\Omega})$.

We also have to show that $\gamma_0^{a-1} K_D = I$. For functions u on Ω , $\gamma_0^{a-1} u = \Gamma(a) \gamma_0(d^{1-a} u)$, where $d(x) = \text{dist}(x, \partial\Omega)$. With our special choice of local coordinates, this carries over from $U_i \cap \Omega$ to $V_i \cap \mathbb{R}_+^n$ as $\Gamma(a) \gamma_0(x_n^{1-a} \underline{u})$ (the trace at $x_n = 0$), when $\text{supp } u \subset U_i$. When u

is multiplied by a function ζ , $\gamma_0^{a-1}(\zeta u) = \gamma_0(\zeta)\gamma_0^{a-1}(u)$, which carries over as $\gamma_0(\underline{\zeta})\gamma_0^{a-1}(\underline{u})$. Recall from (5.25) (or (A.15)) that the boundary value $\gamma_0^{a-1}(\underline{u})$ can also be described as $\gamma_0(\Xi_+^{a-1}\underline{u})$ or $\gamma_0(\Lambda_+^{a-1}\underline{u})$.

Let φ be given as above. Write $\varphi = \sum_{0 \leq i \leq I_0} \varrho_i \varphi$, then

$$\gamma_0^{a-1} K_D \varphi = \sum_i \gamma_0^{a-1} (\underline{\zeta}_i^1 \underline{K}_D^{(i)} \underline{\varrho}_i \varphi) \circ \kappa_i.$$

Let us consider the i 'th piece, carried over to V_i :

$$\begin{aligned} \gamma_0^{a-1} (\underline{\zeta}_i^1 \underline{K}_D^{(i)} \underline{\varrho}_i \varphi) &= \gamma_0(\underline{\zeta}_i^1) \gamma_0^{a-1} (\underline{K}_D^{(i)} \underline{\varrho}_i \varphi) \\ &= \gamma_0(\underline{\zeta}_i^1) \gamma_0^{a-1} (\Lambda_+^{1-a} e^+ r^+ \Lambda_+^{-1} \tilde{Q}_1^{(i)+} (\underline{\varrho}_i \varphi \otimes \delta)) \\ &= \gamma_0(\underline{\zeta}_i^1) \gamma_0(r^+ \Lambda_+^{-1} \tilde{Q}_1^{(i)+} (\underline{\varrho}_i \varphi \otimes \delta)) \\ &= \gamma_0(\underline{\zeta}_i^1) \underline{\varrho}_i \varphi = \underline{\varrho}_i \varphi. \end{aligned}$$

In the step leading to the last line, we used that $\tilde{Q}_1^{(i)+}$ has a symbol $1 + f^+$, where $f^+ \in S^0(\mathcal{H}^+)$ and hence does not contribute to the boundary value, so only K_0 remains (as in the proof of Theorem 5.2).

Carrying the formulas back to the U_i and summing over i , we obtain the conclusion

$$(5.35) \quad \gamma_0^{a-1} K_D \varphi = \varphi.$$

The last part of the proof goes in the same way as in Corollary 5.3. \square

Remark 5.5. In the above proof, one can replace K_D by the operator defined using the family Ξ_\pm^t instead of Λ_\pm^t in the local coordinate systems; it then gets the form

$$(5.36) \quad K_D \varphi = \sum_{i=1}^{I_0} (\underline{\zeta}_i^1 \Xi_+^{1-a} e^+ K_{\underline{Q}_+}^{(i)} \underline{\varrho}_i \varphi) \circ \kappa_i, \text{ where } K_{\underline{Q}_+}^{(i)} \underline{\varrho}_i \varphi = r^+ \Xi_+^{-1} \tilde{Q}^{(i)+} (\underline{\varrho}_i \varphi \otimes \delta(x_n)).$$

The calculations go as above, except that in the treatment of the term corresponding to (5.33), $\Xi_+^a (1 - \underline{\zeta}_i^1) \Xi_+^{-a}$ is only a generalized ψ do; but it can be checked to have a symbol of the form of a function plus a term in $S^0(\mathcal{H}^+)$ (only integer powers of Ξ_+^1 appear in the terms calculated by the Leibniz product formula), and the Poisson operator construction goes through.

There is a certain “uniqueness modulo smoothing operators”:

Corollary 5.6. *In the situation of Theorem 5.4, if ${}^t(r^+ P - \gamma_0^{a-1})$ has the inverse $(R_D - K_D)$, and $K_{D,1}$ is a Poisson-like operator constructed as in Theorem 5.4 or Remark 5.5, then*

$$(5.37) \quad K_D - K_{D,1}: H^{s-a+\frac{1}{2}}(\partial\Omega) \rightarrow \mathcal{E}_a(\overline{\Omega}).$$

Proof. We have that

$$\begin{aligned}
(0 \quad K_D - K_{D,1}) &= (R_D \quad K_D) - (R_D \quad K_{D,1}) \\
&= (R_D \quad K_D) \left[\begin{pmatrix} r^+ P \\ \gamma_0^{a-1} \end{pmatrix} (R_D \quad K_{D,1}) - \begin{pmatrix} \mathcal{S}_1 & \mathcal{S} \\ 0 & 0 \end{pmatrix} \right] \\
&\quad - (R_D \quad K_D) \begin{pmatrix} r^+ P \\ \gamma_0^{a-1} \end{pmatrix} (R_D \quad K_{D,1}) \\
&= (R_D \mathcal{S}_1 \quad R_D \mathcal{S}).
\end{aligned}$$

Since $R_D: C^\infty(\overline{\Omega}) \rightarrow \mathcal{E}_a(\Omega)$, $K_D - K_{D,1} = R_D \mathcal{S}$ satisfies (5.37). \square

6. The Dirichlet-to-Neumann operator, elliptic Neumann problems.

We can now calculate the *Dirichlet-to-Neumann* operator

$$(6.1) \quad S_{DN} = \gamma_1^{a-1} K_D,$$

finding its symbol in local coordinates. In particular, we determine its principal symbol, which gives a criterion for ellipticity of the Neumann problem.

Theorem 6.1. *1° For P considered on $\overline{\mathbb{R}}_+^n$ as in Theorem 5.2, with K_D defined by (5.16–19), S_{DN} is the first-order ψ do $S_{DN} = \text{OP}'(s_{DN}(x', \xi'))$ in \mathbb{R}^{n-1} with symbol (cf. (5.10))*

$$\begin{aligned}
s_{DN}(x', \xi') &= -s_{Q^+}(x', \xi') - a\langle \xi' \rangle, \text{ where} \\
(6.2) \quad s_{Q^+}(x', \xi') &= \frac{1}{2\pi} \int^+ h^+ q^+(x', 0, \xi) d\xi_n = \lim_{z_n \rightarrow 0^+} \mathcal{F}_{\xi_n \rightarrow z_n}^{-1} h^+ q^+(x', 0, \xi).
\end{aligned}$$

In particular, if $P = (1 - \Delta)^a$, then $S_{DN} = -a\langle D' \rangle$.

2° For P considered on $\overline{\Omega}$ as in Theorem 5.4, with K_D defined by (5.36), S_{DN} is the first-order ψ do on $\partial\Omega$ described by

$$(6.3) \quad S_{DN}\varphi = \sum_{i=1}^{I_0} (\gamma_0(\underline{\zeta}_i^1) \text{OP}'(-s_{\underline{Q}^{(i)+}}(x', \xi') - a\langle \xi' \rangle) \underline{\varrho}_i \varphi) \circ \kappa_i.$$

where $s_{\underline{Q}^{(i)+}}(x', \xi')$ is constructed from the symbol of $\underline{Q}^{(i)+}$ as in (6.2).

Proof. 1°. Recall from (3.8) that $\gamma_1^{a-1}u = \gamma_0 \partial_n \Xi_+^{a-1}u - (a-1)\langle D' \rangle \gamma_0^{a-1}u$. We have immediately for the second term:

$$(6.4) \quad -(a-1)\langle D' \rangle \gamma_0^{a-1}K_D = -(a-1)\langle D' \rangle,$$

by (5.21). For the first term we apply composition rules from the Boutet de Monvel calculus:

$$\begin{aligned}
\gamma_0 \partial_n \Xi_+^{a-1}K_D &= \gamma_0 \partial_n \Xi_+^{a-1} \Xi_+^{1-a} e^+ \text{OPK}(\chi_+^{-1}(\xi) \# \tilde{q}^{+'}(x', 0, \xi)) \\
&= \gamma_0 \text{OPK}(i\xi_n(\langle \xi' \rangle + i\xi_n)^{-1} \# \tilde{q}^{+'}(x', 0, \xi)).
\end{aligned}$$

This is the ψ do with symbol

$$\frac{1}{2\pi} \int^+ \frac{i\xi_n}{\langle \xi' \rangle + i\xi_n} \# \tilde{q}^{+'} d\xi_n = \frac{1}{2\pi} \int^+ \left(1 - \frac{\langle \xi' \rangle}{\langle \xi' \rangle + i\xi_n}\right) \# (1 + h^+ \tilde{q}^{+'}) d\xi_n = -\langle \xi' \rangle + \frac{1}{2\pi} \int^+ h^+ \tilde{q}^{+'} d\xi_n.$$

We have here used that $\int^+ 1 d\xi_n = 0$, $\frac{1}{2\pi} \int^+ (\langle \xi' \rangle + i\xi_n)^{-1} d\xi_n = 1$, and that the plus-integral vanishes on functions in \mathcal{H}^+ that are $O(\xi_n^{-2})$, hence on $(\langle \xi' \rangle + i\xi_n)^{-1} \# h^+ \tilde{q}^{+'}$.

The two terms together give the ψ do with symbol

$$-\langle \xi' \rangle + \frac{1}{2\pi} \int^+ h^+ \tilde{q}^{+'} d\xi_n - (a-1)\langle \xi' \rangle = \frac{1}{2\pi} \int^+ h^+ \tilde{q}^{+'} d\xi_n - a\langle \xi' \rangle.$$

There is a further reduction of the plus-integral. We know that $q^+(x, \xi)$ is of the form $q^+ = 1 + f^+$, where $f^+ = h^+ q^+$ lies in $S^0(\mathcal{H}^+)$. The parametrix \tilde{q}^+ of q^+ has the expansion

$$\tilde{q}^+ \sim 1 - f^+ + f^+ \# f^+ - \dots + (-1)^k (f^+)^{\#k} + \dots,$$

where $(-1)^k (f^+)^{\#k} \in S^0(\mathcal{H}^+ \cap \mathcal{H}_{-2})$ for $k \geq 2$ (all terms have at least two factors in \mathcal{H}^+), so we can assume that \tilde{q}^+ is of the form

$$\tilde{q}^+ = 1 - f^+ + r, \quad r \in S^0(\mathcal{H}^+ \cap \mathcal{H}_{-2}).$$

Then

$$\int^+ h^+ \tilde{q}^+ d\xi_n = \int^+ \tilde{q}^+ d\xi_n = \int^+ (1 - f^+ + r) d\xi_n = - \int^+ f^+ d\xi_n = - \int^+ h^+ q^+ d\xi_n.$$

Now it is actually the (x', y_n) -form of the symbol of \tilde{Q}^+ that is used instead of the x -form, cf. (A.11), but this does not change the value, as we shall now show:

The preceding calculations are true also for $\tilde{q}^{+'}$ and $q^{+'}$, so we arrive at having to calculate $\frac{1}{2\pi} \int^+ h^+ q^{+'}(x', 0, \xi) d\xi_n$. As in the proof details after (4.15), we can use the observation from [G90] Lemma 10.18 that for a function $\varphi(\xi_n) \in \mathcal{H}^+$, $\frac{1}{2\pi} \int^+ \varphi(x_n) d\xi_n = \lim_{z_n \rightarrow 0+} [\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \varphi](z_n)$. Since

$$[\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} h^+ q^{+'}(x', 0, \xi)](z_n) \sim \sum_{j \in \mathbb{N}_0} \frac{1}{j!} z_n^j [\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} h^+ \partial_{x_n}^j q^+(x', 0, \xi)](z_n),$$

the limits for $z_n \rightarrow 0+$ satisfy

$$\lim_{z_n \rightarrow 0+} [\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} h^+ q^{+'}(x', 0, \xi)](z_n) = \lim_{z_n \rightarrow 0+} [\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} h^+ q^+(x', 0, \xi)](z_n).$$

Thus $h^+ q^{+'}$ gives the same value as $h^+ q^+$ in the plus-integral, and formula (6.2) follows.

When $P = (1 - \Delta)^a$, then $Q = I$ with $h^+ \tilde{q}^+ = 0$, so only the term $-a\langle \xi' \rangle$ remains. The formula was shown in this case in [G14], Appendix.

2°. We can assume that the ζ_i^1 are constant in x_n for small x_n , then

$$\gamma_1^{a-1} \underline{u} = \Gamma(a+1) \gamma_0(\partial_n(\zeta_i^1 \underline{u}/x_n^{a-1})) = \Gamma(a+1) \gamma_0(\zeta_i^1) \gamma_0(\partial_n(\underline{u}/x_n^{a-1})) = \gamma_0(\zeta_i^1) \gamma_1^{a-1}(\underline{u}).$$

Now

$$(6.5) \quad \gamma_1^{a-1} K_D \varphi = \sum_i (\gamma_0(\zeta_i^1) \gamma_1^{a-1} (\Xi_+^{1-a} e^+ K_{\underline{Q}^+}^{(i)} \underline{\varphi})) \circ \kappa_i,$$

where each term is calculated as under 1°. This leads to (6.3). \square

Theorem 6.2. *Hypotheses as in Theorem 6.1.*

The principal symbol of S_{DN} is

$$(6.6) \quad s_{DN,0}(x', \xi') = -\frac{1}{2\pi} \int^+ h^+ q_0^+(x', 0, \xi) d\xi_n - a|\xi'|,$$

where q_0^+ is the plus-factor in the principal symbol $q_0(x, \xi) = s_0^{-1} p_0(x, \xi) |\xi|^{-2a}$, constructed as in [G16], Th. 2.6, in the local coordinates used in Theorem 6.1. Here

$$(6.7) \quad \frac{1}{2\pi} \int^+ h^+ q_0^+(x', 0, \xi) d\xi_n = \lim_{z_n \rightarrow 0+} \mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \log q_0(x', 0, \xi).$$

The Neumann problem defined for $u \in H^{(a-1)(s)}(\overline{\Omega})$, $s > a + \frac{1}{2}$,

$$(6.8) \quad r^+ P u = f \text{ on } \Omega, \quad \gamma_1^{a-1} u = \psi,$$

has a parametrix

$$(6.9) \quad (R_N \quad K_N) : \overline{H}^{s-2a}(\Omega) \times H^{s-a-\frac{1}{2}}(\partial\Omega) \rightarrow H^{(a-1)(s)}(\overline{\Omega}),$$

if and only $s_{DN}(x', \xi')$ is nonvanishing for $\xi' \neq 0$, i.e., S_{DN} is elliptic. In the affirmative case, a parametrix is

$$(6.10) \quad (R_N \quad K_N) = ((I - K_D \tilde{S}_{DN} \gamma_1^{a-1}) R_D \quad K_D \tilde{S}_{DN}),$$

where \tilde{S}_{DN} is a parametrix of S_{DN} . The Neumann problem is then said to be elliptic.

Proof. It is known that the principal symbol of a ψ do on a manifold (in this case $\partial\Omega$) is an invariant function of the cotangent variables (x', ξ') , so we just need to indicate a way to find it; this goes via the localization (6.3), when we moreover use that $\zeta_i^1 \varrho_i = \varrho_i$ for each i . Then the formula

$$s_{DN,0}(x', \xi') = -\frac{1}{2\pi} \int^+ h^+ q_0^+(x', 0, \xi) d\xi_n - a|\xi'|,$$

follows from Theorem 6.1.

For the description in (6.7), we recall from the proof of Theorem 2.6 in [G16] that $q_0^+ = \exp \psi_+$, where $\psi_+ = h^+ \psi$ with $\psi = \log q_0$. Here q_0^+ has the expansion

$$q_0^+ = 1 + \psi_+ + \sum_{k \geq 2} \frac{1}{k!} \psi_+^k,$$

where the sum over $k \geq 2$ is $O(\langle \xi_n \rangle^{-2})$, so this, as well as the term 1, gives 0 when plus-integrals are calculated, and only ψ_+ remains. Thus

$$\frac{1}{2\pi} \int^+ h^+ q_0^+ dx_n = \frac{1}{2\pi} \int^+ \psi_+ dx_n = \lim_{z_n \rightarrow 0+} \mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \psi_+ = \lim_{z_n \rightarrow 0+} \mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \psi,$$

since $\psi_- = \psi - \psi_+ = h^- \psi$ has $\mathcal{F}_{\xi_n \rightarrow z_n}^{-1} \psi_-$ supported in $\overline{\mathbb{R}}_-$.

For the second statement, it is easily checked that if S_{DN} is elliptic with a parametrix \tilde{S}_{DN} , then a parametrix for (6.8) can be constructed from the parametrix for the Dirichlet problem by the formula (6.9). (Similar considerations are carried out in [G14], Sect. 4B.) Conversely, if the Neumann problem has a parametrix with the asserted mapping property, then there is a Fredholm mapping from $\varphi_1 = \gamma_1^{a-1} u$ to $\varphi_0 = \gamma_0^{a-1} u$ for solutions u to the Neumann resp. Dirichlet problem with the same value of $r^+ P u$. This gives a parametrix of S_{DN} ; hence it is elliptic. \square

Example 6.3. To illustrate some of the formulas, let us consider a very simple example with an easy factorization. Let $P = (\beta|D'|^2 + D_n^2 + \lambda)^a$ with $\beta > 0$, $\lambda \geq 0$, in the situation $\mathbb{R}_+^n \subset \mathbb{R}^n$. Then

$$Q = (\beta|D'|^2 + D_n^2 + \lambda)^a(1 - \Delta)^{-a} = \text{OP}(q^- q^+), \quad q^\pm = \left(\frac{(\beta|\xi'|^2 + \lambda)^{\frac{1}{2}} \pm i\xi_n}{\langle \xi' \rangle \pm i\xi_n} \right)^a.$$

Here q^+ has the expansion

$$\begin{aligned} q^+ &= \left(\frac{(\beta|\xi'|^2 + \lambda)^{\frac{1}{2}} + i\xi_n}{\langle \xi' \rangle + i\xi_n} \right)^a = \left(1 + \frac{(\beta|\xi'|^2 + \lambda)^{\frac{1}{2}} - \langle \xi' \rangle}{\langle \xi' \rangle + i\xi_n} \right)^a \\ &= 1 + a \frac{(\beta|\xi'|^2 + \lambda)^{\frac{1}{2}} - \langle \xi' \rangle}{\langle \xi' \rangle + i\xi_n} + O(\langle \xi_n \rangle^{-2}), \end{aligned}$$

so

$$\begin{aligned} -\frac{1}{2\pi} \int^+ h^+ q^+ d\xi_n &= -a((\beta|\xi'|^2 + \lambda)^{\frac{1}{2}} - \langle \xi' \rangle) \frac{1}{2\pi} \int^+ \frac{1}{\langle \xi' \rangle + i\xi_n} d\xi_n \\ &= -a(\beta|\xi'|^2 + \lambda)^{\frac{1}{2}} + a\langle \xi' \rangle. \end{aligned}$$

Then

$$(6.10) \quad s_{DN} = -a(\beta|\xi'|^2 + \lambda)^{\frac{1}{2}} + a\langle \xi' \rangle - a\langle \xi' \rangle = -a(\beta|\xi'|^2 + \lambda)^{\frac{1}{2}}, \quad s_{DN,0} = -a\beta^{\frac{1}{2}}|\xi'|.$$

In this case s_{DN} is elliptic for all choices of $\beta > 0$, $\lambda \geq 0$, and the Neumann problems for these operators are elliptic. If λ is replaced by a nonnegative function $V(x)$, the calculations are valid on the principal symbol level.

In the paper [G15] the continuity properties of ${}^t(r^+ P \gamma_0^{a-1})$ and its parametrix $(R_D \ K_D)$ were shown in H_p^s -Sobolev spaces (essentially, see Ths. 4.2, 5.1 and 6.5 there), and in [G14], which was written after [G15], they were extended to large families of Besov and Triebel-Lizorkin spaces, including results for elliptic Neumann problems (see Ths. 3.2, 3.5 and 4.3 there). Only the structure of K_D and K_N in the case of a curved domain was not explained in detail. We now have the full explanation above of how K_D and K_N consist of operators from the Boutet de Monvel calculus composed with operators Ξ_+^{1-a} or Λ_+^{1-a} in local coordinates, so the mapping properties extend readily from the H_2^s -scales to the H_p^s -scales as in [G15], and to the Besov and Triebel-Lizorkin scales as in [G14] Sect. 3 (based on Johnsen [J96] and its references). We can therefore conclude from [G15, G14] the following formulations of the mapping properties of the parametrices, for smooth bounded domains Ω . (We recall that $H_p^s = F_{p,2}^s$ and $C_*^s = B_{\infty,\infty}^s$.)

Corollary 6.4. *Let $1 \leq p \leq \infty$ and $0 < q \leq \infty$ (with $p < \infty$ in the F -cases, (6.11), (6.13) and the first line of (6.15) below).*

The parametrix of the Dirichlet problem maps as follows, for $s > a - 1/p'$:

$$(6.11) \quad \begin{aligned} (R_D \ K_D) : \overline{F}_{p,q}^{s-2a}(\Omega) \times B_{p,p}^{s-a+1/p'}(\partial\Omega) &\rightarrow F_{p,q}^{(a-1)(s)}(\overline{\Omega}), \text{ in particular} \\ (R_D \ K_D) : \overline{H}_p^{s-2a}(\Omega) \times B_{p,p}^{s-a+1/p'}(\partial\Omega) &\rightarrow H_p^{(a-1)(s)}(\overline{\Omega}). \end{aligned}$$

Moreover, for $s > a - 1/p'$, $t > a - 1$,

$$(6.12) \quad \begin{aligned} (R_D \quad K_D): \overline{B}_{p,q}^{s-2a}(\Omega) \times B_{p,q}^{s-a+1/p'}(\partial\Omega) &\rightarrow B_{p,q}^{(a-1)(s)}(\overline{\Omega}), \text{ in particular} \\ (R_D \quad K_D): \overline{C}_*^{t-2a}(\Omega) \times C_*^{t-a+1}(\partial\Omega) &\rightarrow C_*^{(a-1)(t)}(\overline{\Omega}). \end{aligned}$$

The parametrix of the Neumann problem (when elliptic) maps as follows, for $s > a+1/p$:

$$(6.13) \quad \begin{aligned} (R_N \quad K_N): \overline{F}_{p,q}^{s-2a}(\Omega) \times B_{p,p}^{s-a-1/p}(\partial\Omega) &\rightarrow F_{p,q}^{(a-1)(s)}(\overline{\Omega}), \\ (R_N \quad K_N): \overline{H}_p^{s-2a}(\Omega) \times B_{p,p}^{s-a-1/p}(\partial\Omega) &\rightarrow H_p^{(a-1)(s)}(\overline{\Omega}). \end{aligned}$$

Moreover, for $s > a + 1/p$, $t > a$,

$$(6.14) \quad \begin{aligned} (R_N \quad K_N): \overline{B}_{p,q}^{s-2a}(\Omega) \times B_{p,q}^{s-a-1/p}(\partial\Omega) &\rightarrow B_{p,q}^{(a-1)(s)}(\overline{\Omega}), \\ (R_N \quad K_N): \overline{C}_*^{t-2a}(\Omega) \times C_*^{t-a}(\partial\Omega) &\rightarrow C_*^{(a-1)(t)}(\overline{\Omega}). \end{aligned}$$

Being a ψ do of order 1, the Dirichlet-to-Neumann operator S_{DN} maps as follows, for all $s \in \mathbb{R}$:

$$(6.15) \quad \begin{aligned} S_{DN}: F_{p,q}^s(\partial\Omega) &\rightarrow F_{p,q}^{s-1}(\partial\Omega), \quad S_{DN}: H_p^s(\partial\Omega) \rightarrow H_p^{s-1}(\partial\Omega), \\ S_{DN}: B_{p,q}^s(\partial\Omega) &\rightarrow B_{p,q}^{s-1}(\partial\Omega), \quad S_{DN}: C_*^s(\partial\Omega) \rightarrow C_*^{s-1}(\partial\Omega). \end{aligned}$$

Remark 6.5. In connection with the Neumann trace operator γ_1^{a-1} there are many other meaningful boundary conditions for r^+P , namely conditions of the (Robin-like) form

$$(6.16) \quad \gamma_1^{a-1}u + L\gamma_0^{a-1}u = \psi,$$

where L is a first-order ψ do on $\partial\Omega$; they are *local* when L is a differential operator. The problem for r^+P with condition (6.16) is *elliptic* when $S_{DN} + L$ is elliptic, i.e., in local coordinates, $s_{DN,0}(x', \xi') + l_0(x', \xi') \neq 0$ for $\xi' \neq 0$. Then the problem is Fredholm solvable in scales of spaces as in Theorem 6.2 and Corollary 6.4.

In particular, when P is principally like $(-\Delta)^a$ (i.e., $P - (-\Delta)^a$ is of order $2a - 1$), we have, when L in local coordinates at the boundary has the principal part $\sum_{j=1}^{n-1} b_j(x')\partial_j$, that ellipticity holds when $-a|\xi'| + b(x') \cdot i\xi'$ is nonvanishing for $\xi' \neq 0$. This is satisfied when the vector b is real; for complex b it holds when $|\operatorname{Im} b(x')| < a$ for all x' .

Finally, some remarks on nonsmooth situations: Abels [A05a, A05b] has established a nonsmooth version of results from the Boutet de Monvel calculus in H_p^s and $B_{p,q}^s$ -spaces for operators with for example Hölder continuous x -dependence. This is based on earlier works on nonsmooth ψ do's, in particular Marschall [M88]. In this generalization, the range of possible s is limited to specific bounded intervals. Certainly, much of the above can be extended to such operators, when Ω is smooth. To allow nonsmoothness of Ω , one would need to handle nonsmooth coordinate changes, but here the results are scarce. — Another strategy would be to use the results known in smooth cases in combination with perturbation arguments (for example, Vishik and Eskin [E81] used such techniques to pass

from constant-coefficient cases to variable coefficients). Much remains to be investigated for such questions.

The results from the smooth studies can serve as a model for what one would want to show in nonsmooth cases. In comparison, many results through the years involving the Laplacian $-\Delta$ are generalizations of old and well-established results in smooth cases, for example based on Green's formula. The “old, well-established” results have not been available in the case of the fractional Laplacian $(-\Delta)^a$; the present strivings with ψ do methods help to fill that gap.

Appendix.

We here collect some notation and results from the Boutet de Monvel calculus [B71, G90, G96, G09] and from [G16], that are used in the text.

The operators below are defined for the situation $\mathbb{R}_+^n \subset \mathbb{R}^n$. Besides pseudodifferential operators (ψ do's) on \mathbb{R}^n , cf. (2.2), there are *Poisson operators* K , and *trace operators* T of class 0:

$$(A.1) \quad \begin{aligned} (K\varphi)(x) &= \text{OPK}(k(x', \xi))\varphi = (2\pi)^{-n} \int e^{ix \cdot \xi} k(x', \xi) \hat{\varphi}(\xi') d\xi \\ (Tu)(x') &= \text{OPT}(t(x', \xi))u = (2\pi)^{-n} \int e^{ix' \cdot \xi'} t(x', \xi) \widehat{e^+ u} d\xi, \end{aligned}$$

with suitable interpretations of the integrals, and ψ do's $S = \text{OP}'(s(x', \xi'))$ on \mathbb{R}^{n-1} defined as in (2.2). In general, the trace operators are supplied with standard trace operators $\sum_{0 \leq j < r} S_j \gamma_j$ with ψ do coefficients S_j ; then they are of class r . (We shall not need *singular Green operators* in the present paper.) The symbols in (A.1) behave w.r.t. ξ_n as elements of the spaces \mathcal{H}^+ resp. \mathcal{H}_{-1}^- :

We recall that with $d \in \mathbb{Z}$, \mathcal{H}_d denotes the space of C^∞ -functions $f(t)$ on \mathbb{R} such that $k(\tau) = \tau^d f(\tau^{-1})$ coincides with a C^∞ -function for $-1 < \tau < 1$ (i.e., the derivatives of f match in a good way for $t \rightarrow \pm\infty$). Here

$$(A.2) \quad \mathcal{H}_{-1} = \mathcal{F}(e^- \mathcal{S}_- \oplus e^+ \mathcal{S}_+), \quad \mathcal{H}_d = \mathcal{H}_{-1} \oplus \mathbb{C}_d[t] \text{ for } d \geq 0,$$

where $\mathcal{S}_\pm = r^\pm \mathcal{S}(\mathbb{R}) = \mathcal{S}(\overline{\mathbb{R}}_\pm)$, and $\mathbb{C}_d[t]$ stands for the space of polynomials of degree $\leq d$ in t . We also denote

$$\mathcal{H} = \bigcup_d \mathcal{H}_d = \mathcal{H}_{-1} \oplus \mathbb{C}[t], \quad \mathbb{C}[t] = \bigcup_d \mathbb{C}_d[t].$$

With a slight asymmetry, one defines

$$(A.3) \quad \begin{aligned} \mathcal{H}^+ &= \mathcal{F}(e^+ \mathcal{S}_+), \quad \mathcal{H}^- = \mathcal{F}(e^- \mathcal{S}_-) \oplus \mathbb{C}[t], \quad \mathcal{H}_d^- = \mathcal{F}(e^- \mathcal{S}_-) \oplus \mathbb{C}_d[t], \text{ whereby} \\ \mathcal{H} &= \mathcal{H}^+ \oplus \mathcal{H}^-, \quad \mathcal{H}_d = \mathcal{H}^+ \oplus \mathcal{H}_d^-, \text{ for } d \geq -1; \text{ in particular } \mathcal{H}_{-1} = \mathcal{H}^+ \oplus \mathcal{H}_{-1}^-. \end{aligned}$$

The mappings h^\pm are defined on \mathcal{H} such that they are complementing projections with ranges

$$(A.4) \quad h^+ \mathcal{H} = \mathcal{H}^+, \quad h^- \mathcal{H} = \mathcal{H}^-, \text{ in particular, } h^- \mathcal{H}_d = \mathcal{H}_d^- \text{ for } d \geq -1.$$

The symbol h_{-1} denotes the projection from \mathcal{H} to \mathcal{H}_{-1} that removes the polynomial part. The space \mathcal{H}_{-1}^- equals the space of conjugates of functions in \mathcal{H}^+ . \mathcal{H}^+ can also be denoted \mathcal{H}_{-1}^+ when convenient. When $f \in \mathcal{H}_{-1}$, $\overline{h^- f} = h^+(\overline{f})$.

The symbol spaces for Poisson resp. trace operators are the spaces $S^m(\mathcal{H}^+)$ resp. $S^m(\mathcal{H}^-)$. Here for example $S^m(\mathcal{H}^+) = S_{1,0}^m(\mathbb{R}^m, \mathbb{R}^{n-1}, \mathcal{H}^+)$ consists of functions $f(X, \xi', \xi_n)$ that are in \mathcal{H}^+ w.r.t. ξ_n and satisfy the estimates

$$(A.5) \quad \|D_X^\beta D_{\xi'}^\alpha D_{\xi_n}^k h_{-1}(\xi_n^{k'} f(X, \xi', \xi_n))\|_{L_2(\mathbb{R})} \leq C_{\alpha, \beta, k, k'} \langle \xi' \rangle^{m + \frac{1}{2} - k + k' - |\alpha|},$$

for all indices. The symbols considered here moreover have asymptotic expansions in terms f_j that are homogeneous in (ξ', ξ_n) of degree $m - j$ for $|\xi'| \geq 1$, such that $f - \sum_{0 \leq j < J} f_j$ is in $S^{m-J}(\mathcal{H}^+)$ for all J . The definitions with \mathcal{H}^+ replaced by \mathcal{H}_{-1}^- or \mathcal{H}_{-1} are similar, and one can also define spaces with \mathcal{H}^+ replaced by \mathcal{S}_+ etc., cf. [G16].

It follows from [G16] Th. 2.6, that when P is a classical ψ do of order $2a$, even (cf. (1.2)), and elliptic avoiding a ray, then the principal symbol $q_0(x, \xi) = p_0(x, \xi)[\xi]^{-2a}$ of $Q = \Xi_-^{-a} P \Xi_+^{-a}$ has a factorization

$$(A.6) \quad q_0(x, \xi) = q_0^-(x, \xi) q_0^+(x, \xi),$$

where $q_0^+ = 1 + f^+$ with $f^+ \in S_{1,0}^0(\mathbb{R}^n, \mathbb{R}^{n-1}, \mathcal{H}^+)$, and $q_0^- = s_0 + f^-$ with $f^- \in S_{1,0}^0(\mathbb{R}^n, \mathbb{R}^{n-1}, \mathcal{H}_{-1}^-)$; we have here included $s_0(x) = p_0(x, 0, 1)$ as a factor in q_0^- .

The functions q_0^\pm are generalized ψ do symbols, like $([\xi'] \pm i\xi_n)^d$, and therefore define operators with have slightly weaker mapping properties than ψ do's (see [G16] Sect. 2), but when Poisson- and trace operators are constructed from such symbols (see further below), they enter in the same way as ψ do symbols with the transmission property, because it is the \mathcal{H}_{-1} -property that is used.

There is also a factorization construction involving the true ψ do families Λ_\pm^t . Here we define

$$(A.7) \quad Q_1 = \Lambda_-^{-a} P \Lambda_+^{-a}, \quad q_1(x, \xi) = \lambda_-^{-a} \# p(x, \xi) \lambda_+^{-a},$$

cf. (2.3). The principal symbol of Q_1 is again q_0 , that factorizes as in (A.6). By [G16] Th. 2.7 there is a factorization of the full symbol $q_1(x, \xi)$, leading to a factorization of Q_1 , hence of P :

$$(A.8) \quad q_1 \sim q_1^- \# q_1^+, \quad Q_1 \sim Q_1^- Q_1^+, \quad P \sim \Lambda_-^a Q_1^- Q_1^+ \Lambda_+^a,$$

where $Q_1^\pm = \text{OP}(q_1^\pm)$ (also here, s_0 is included as a factor to the left in q_1^-).

There is a similar full factorization in terms of $Q = \Xi_-^{-a} P \Xi_+^{-a}$. Th. 2.7 of [G16] does not apply directly to $\chi_-^{-a} \# p(x, \xi) \chi_+^{-a}$ since this already passes outside the true ψ do's, but the proof can be adapted to it since it is the \mathcal{H}_{-1} -properties that are used. Alternatively, we can use that $\Lambda_\pm^\mu = \Xi_\pm^\mu (I + \Psi_\pm^\mu)$, where Ψ_\pm^μ have symbols $\psi_\pm^\mu(\xi)$ in $S^0(\mathcal{H}_{-1}^\pm)$, cf. [G15], (1.16)ff. and Lemma 6.6. This gives by insertion in (A.8), taking $\mu = a$:

$$(A.9) \quad \begin{aligned} P &\sim \Xi_-^a (1 + \Psi_-^a) Q_1^- Q_1^+ (1 + \Psi_+^a) \Xi_-^a = \Xi_-^a Q^- Q^+ \Xi_-^a, \text{ where} \\ Q^- &= (1 + \Psi_-^a) Q_1^-, \quad Q^+ = Q_1^+ (1 + \Psi_+^a). \end{aligned}$$

Here Q^- is of the form $s_0 + F^-$ where F^- has symbol in $S^0(\mathcal{H}_{-1}^-)$, and $Q^+ = 1 + F^+$ where F^+ has symbol in $S^0(\mathcal{H}^+)$.

Since the symbols q_1^-, q_1^+, q^-, q^+ have invertible principal symbols, they have full parametrix symbols $\tilde{q}_1^-, \tilde{q}_1^+, \tilde{q}^-, \tilde{q}^+$ (as in [G16] Th. 2.9); hereby Q_1^-, Q_1^+, Q^-, Q^+ have parametrices $\tilde{Q}_1^-, \tilde{Q}_1^+, \tilde{Q}^-, \tilde{Q}^+$.

Pseudodifferential operators defined from a symbol $p(x, \xi)$ as in (2.2) are a special case of ψ do's defined from amplitude functions $r(x, y, \xi)$, also called symbols “in (x, y) -form”, by

$$(A.10) \quad Ru = \text{OP}(r(x, y, \xi))u = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} r(x, y, \xi) u(y) dy d\xi.$$

Symbols $p(x, \xi)$ used as in (2.2) are then said to be “in x -form”, and there is an asymptotic formula for the x -form symbol producing the same operator as the symbol $r(x, y, \xi)$ in (A.10) (see e.g. [G09] Th. 7.13). A useful special case is when $r(x, y, \xi)$ in (A.10) is independent of x_n and y' (which can then be left out); then it is said to be “in (x', y_n) -form”. The transition from x -form to (x', y_n) -form goes as follows: An operator $R = \text{OP}(r(x, \xi))$ has a symbol r' in (x', y_n) -form related to the symbol r in x -form by

$$(A.11) \quad r'(x', y_n, \xi) \sim \sum_{j \in \mathbb{N}_0} \frac{1}{j!} (i\partial_{\xi_n})^j \partial_{x_n}^j r(x', x_n, \xi)|_{x_n=y_n}.$$

We now recall explicitly some rules from the Boutet de Monvel calculus that we need, cf. e.g. [G09], (10.22), Ths. 10.24 and 10.25:

Trace operators arise in particular from compositions $\gamma_0 r^+ R e^+$, where R is a ψ do having the 0-transmission property. When $R = \text{OP}(r(x, \xi))$, then $\gamma_0 r^+ R e^+$ is the trace operator with symbol $h^- r(x', 0, \xi)$,

$$(A.12) \quad \gamma_0 r^+ \text{OP}(r(x, \xi)) e^+ = \text{OPT}(h^- r(x', 0, \xi)).$$

The composition $\varphi \mapsto r^+ R(\varphi(x') \otimes \delta(x_n))$ gives rise to a Poisson operator. It has the simplest expression when R is in (x', y_n) -form, namely

$$(A.13) \quad r^+ \text{OP}(r'(x', y_n, \xi))(\varphi(x') \otimes \delta(x_n)) = \text{OPK}(h^+ r'(x', 0, \xi))\varphi.$$

These rules extend to the generalized ψ do's with symbols in $S^m(\mathcal{H}_{-1})$, m integer ≤ 0 , since it is the property of the symbol being in \mathcal{H}_{-1} with respect to ξ_n that is used in the proofs of (A.12), (A.13).

Further composition rules in the calculus are amply described in [G09] Sect. 10.4. Let us just recall that when $T = \text{OPT}(t(x', \xi))$ and $K = \text{OPK}(k(x', \xi))$, then $S = TK$ is a ψ do $S = \text{OP}'(s(x', \xi'))$ on \mathbb{R}^{n-1} with symbol

$$(A.14) \quad s(x', \xi') = \frac{1}{2\pi} \int^+ t(x', \xi) \# k(x', \xi) d\xi_n.$$

Here the *plus-integral* $\int^+ f(t) dt$ is a linear extension of, on one hand, the ordinary integral on \mathbb{R} of functions $f \in L_1(\mathbb{R})$ and, on the other hand, the integral over a curve encircling

the poles in \mathbb{C}_+ when $f(t)$ extends to a meromorphic function of t in \mathbb{C}_+ . The plus-integral vanishes on \mathcal{H}^- . It also vanishes on functions in \mathcal{H}^+ that are $O(t^{-2})$ at infinity, since the integral can then be turned into an integral over a contour in \mathbb{C}_- .

Also for Poisson and trace operators, one can define them in x' -form, (x', y') -form or y' -form. The adjoints of the Poisson operators $\text{OPK}(k(x', \xi))$ are the trace operators $\text{OPT}(\bar{k}(y', \xi))$; the latter are exactly the trace operators of class zero, written in y' -form.

We shall need the result that $\gamma_0^{a-1}u$ is described equally well using Ξ_+^{a-1} or Λ_+^{a-1} . By the remarks before (A.9), $\Lambda_+^{a-1} = (1 + \Psi_+^{a-1})\Xi_+^{a-1}$, where Ψ_+^{a-1} has symbol $\psi_+^{a-1}(\xi) \in S^0(\mathcal{H}^+)$. Let $u \in H^{(a-1)(s)}(\overline{\mathbb{R}_+^n})$, $s > a - \frac{1}{2}$, then $\tilde{u} = r^+\Xi_+^{a-1}u \in \overline{H}^{s-a+1}(\mathbb{R}_+^n)$ with $u = \Xi_+^{1-a}e^+\tilde{u}$ (by definition). We know from [G15], Sect. 5, and it is reproved in Section 2 above, that $\gamma_0^{a-1}u = \gamma_0\Xi_+^{a-1}u$ (as a boundary value from \mathbb{R}_+^n). Then moreover,

$$(A.15) \quad \gamma_0(\Lambda_+^{a-1}u) = \gamma_0((I + \Psi_+^{a-1})\Xi_+^{a-1}u) = \gamma_0(\Xi_+^{a-1}u) = \gamma_0^{a-1}u.$$

Namely, $\gamma_0(\Psi_+^{a-1}\Xi_+^{a-1}u) = \gamma_0(\Psi_+^{a-1}e^+\tilde{u})$ is in the ξ_n -variable calculated as a plus-integral of $\psi_+^{a-1}(\xi)$ multiplied by $\widehat{e^+\tilde{u}}$, where both functions are in \mathcal{H}^+ , hence the product is $O(\langle \xi_n \rangle^{-2})$ and in \mathcal{H}^+ , whereby the plus-integral gives zero. (This kind of calculation also enters in [G16], around (3.31).)

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